Physics for Computer Science Students

Lecture 4: Kinematics of solid bodies and dynamics of material points. Friction

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1 Kinematics of a solid body

There are not ideal rigid bodies in the nature. Nevertheless, the rigid body approach is very useful in modeling the big number of various behaviors of the non-rigid bodies in the cases when the corresponding deformations can be neglected. Such a situation can be observed e.g. in the case of the movements of the planets in the solar system. One can easily neglect the volcano eruption on the Hawaii Island, while this event has a very small influence on the properties of the Plutonium orbit.

The most important feature, which has to be concerned in the motion of the rigid body as compared with the motion of the material point, is the rotational motion. It is obvious that the material point is not spread out, so it is interesting to discuss what new elements of the motion are connected with taking into account the real dimensions of a body.

We say that the rigid body moves in **a pure rotational motion** if the circle is a trajectory of the every point of a body and the centers of these circles lie on a straight line. This line is called **the rotational axis of a body**.

It follows from this definition that in the order to describe the rotational motion it is enough to define the rotational axis and to select one point lying not on the rotational axis (why?).

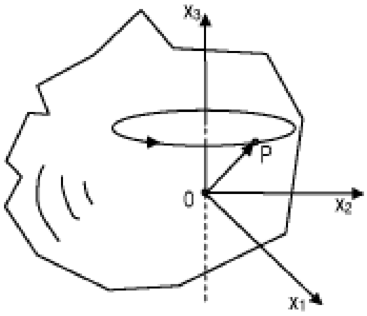

Figure 1: The pure rotational motion of a rigid body around the $0x_3$ axis

In general, the motion of a body consists of the translational motion and of the rotational motion. The rotational motion can take place also around the axis lying outside a body (see e.g. the solar system). In order to describe such a situation it is useful to define in the body the coordinate system rigidly connected with it. Next, one has to learn how to describe the motion of this coordinate system with respect to the coordinate system connected with the fixed reference frame (see Figure 2). Translatory-rotatory movements are in general rather complicated (e.g. see the motion of the Earth with respect to the Sun, and the motion of the Luna with respect to the Earth, and the motion of the Luna with respect to the Sun).

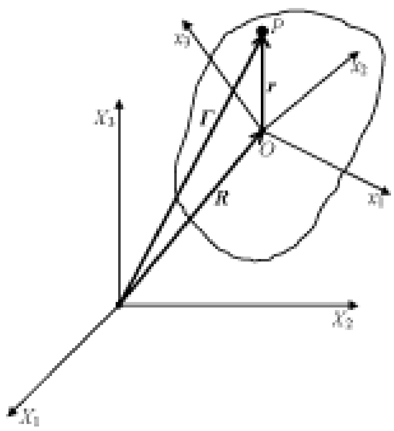

Figure 2: Coordinate systems: movable x_1, x_2, x_3 and fixed X_1, X_2, X_3

We introduce two coordinate systems: fixed coordinate system X_1, X_2, X_3 and movable coordinated system x_1, x_2, x_3 . Movable coordinate system is rigidly connected with the body and participates in all its motions. It is usually placed in the center of the mass of a body.

1.1 Number of degrees of freedom of a rigid body

Let the vector $\boldsymbol{R}$ describes the location of the origin (point O - see Figure 2) of the movable coordinate system. Orientation of the axes of the movable coordinate system with respect to the fixed coordinate system is defined by the three independent angles. Together there are six coordinates: three coordinates of the vector $\boldsymbol{R}$and these three angles.

Conclusion: *every rigid body is a mechanical system with six degrees of freedom.*

Location of an arbitrary point P inside the rigid body with mass m in the fixed coordinate system defines the vector $\boldsymbol{r}$, and in the fixed coordinate system the vector $\boldsymbol{\Gamma}$. The infinitesimal displacement $d\boldsymbol{\Gamma}$ of the point P consists of the translation $d\ \boldsymbol{R}$ of the mass center and of the displacement $d\ \phi\cdot\boldsymbol{r}$ with respect to the mass center, where $d\ \phi$ is the infinitesimal rotation, i.e.

$$d\boldsymbol{\Gamma} = d\ \boldsymbol{R} + d\ \phi\cdot\boldsymbol{r}. \tag{1}$$

1.2 The center of inertia

The momentum of a closed mechanical system has a different form in the different (inertial) systems of reference. If the reference system O' moves with respect to the other reference system O with the velocity $\boldsymbol{V}$, thus in this case the velocities $\boldsymbol{v}'$ and $\boldsymbol{v}$ with respect to these reference systems are connected by the equation

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{V}. \tag{2}$$

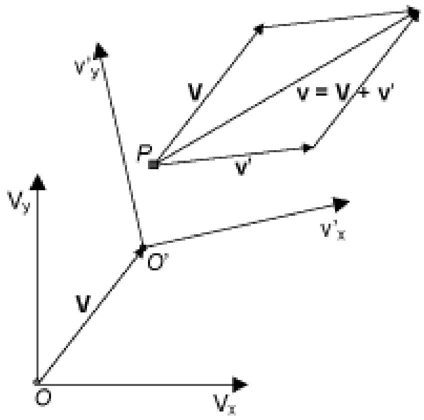

Figure 3: Coordinate systems: movable and coordinate systems; addition of velocities

The rigid body is considered very often as a system of finite or infinite number of material points with the point masses m_i and with the constant distances between them:

$$m = \sum_i m_i,$$

(additivity of masses law). The momentum of a body can be written down as:

$$\boldsymbol{P} = \boldsymbol{p}' + \boldsymbol{V}\sum_i m_i. \tag{3}$$

There exists always such a reference frame K' in which $\boldsymbol{p}' = 0$, thus

$$\boldsymbol{V} = \frac{\boldsymbol{P}}{\sum_i m_i}. \tag{4}$$

If the total momentum of the mechanical system equals zero, it means that the system is fixed with respect to this reference system: the velocity $\boldsymbol{V}$ describes the velocity of a body treated as a whole. The momentum conservation law allows to decide whether the body rests or moves.

Remark: it is favorably to use such a reference system in which the center of inertia of a body rests. In such a way the uniform and straight motion of the considered system is excluded.

Equation (4) can be treated as a derivative with respect to time of the following expression (see Fig. 2)

$$\boldsymbol{R} = \frac{\sum_i m_i \Gamma_i}{\sum_i m_i}. \tag{5}$$

1.3 Velocity

If the motion takes place during the time $d\,t$, thus

$$\frac{d\boldsymbol{\Gamma}}{d\,t} = \boldsymbol{v}, \quad \frac{d\,\boldsymbol{R}}{d\,t} = \boldsymbol{V}, \quad \frac{d\boldsymbol{\phi}}{d\,t} = \boldsymbol{\Omega}, \tag{6}$$

i.e.

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\Omega}\cdot\boldsymbol{r}. \tag{7}$$

1.4 Momentum conservation law

The center of inertia of a body moves uniformly and straight.

2 Energy

2.1 Internal energy

Internal energy E_w - the rest energy of a mechanical system: it consists of the kinetic energy of the relative motion of the particles of a body and the potential energy of their interactions.

2.2 Total energy

Total energy E of the system moving with the velocity $\boldsymbol{V}$ is the sum of its kinetic and internal energies

$$\boldsymbol{E} = \frac{\boldsymbol{m}\ \boldsymbol{V}^2}{2} + \boldsymbol{E}_{\boldsymbol{w}}. \tag{8}$$

3 Moment of momentum

The moment of momentum conservation law follows from the **isotropy of the space** - the Lagrange function should be invariant with respect to the rotations.

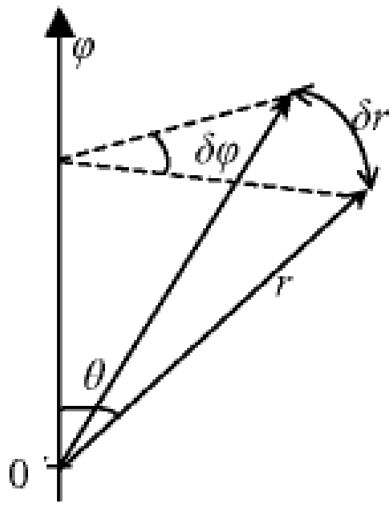

Figure 4: Infinitesimal rotational motion

The increment of the position vector:

$$|\delta\boldsymbol{r}| = r\ \sin\theta\cdot\delta\phi, \quad \delta\boldsymbol{r} = \delta\boldsymbol{\phi}\cdot\boldsymbol{r}. \tag{9}$$

The increment of the velocity:

$$\delta \boldsymbol{v} = \delta\boldsymbol{\phi}\cdot\boldsymbol{v}. \tag{10}$$

Lagrangian:

$$\delta L = \sum_i \left(\frac{\partial L}{\partial r_i}\delta r_i + \frac{\partial L}{\partial v_i}\delta v_i \right) = 0, \tag{11}$$

$$\frac{\partial L}{\partial r_i} = \dot{p}_i, \qquad \frac{\partial L}{\partial v_i} = p_i, \tag{12}$$

Invariance of the Lagrangian

$$\sum_i (\dot{\boldsymbol{p}}_i(\delta\boldsymbol{\phi}\cdot\boldsymbol{r}_i) + p_i(\delta\boldsymbol{\phi}\cdot\boldsymbol{v}_i)) = 0, \tag{13}$$

$$\delta\boldsymbol{\phi}\sum_i (\boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i + \boldsymbol{v}_i \times \boldsymbol{p}_i) = \delta\boldsymbol{\phi}\frac{d}{d\,t}\sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = 0. \tag{14}$$

Moment of momentum:

$$\frac{d}{d\,t}\sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \frac{d}{d\,t}\boldsymbol{M} = 0. \tag{15}$$

This is the **moment of momentum conservation law**.

4 Dynamics of the material point

The material point in the space 3D has three degrees of freedom. It means that it can move on the straight line (one-dimensional motion), on the plane (eg. on the curve like the spatial, ballistic curve, closed curve) on in he absolutely arbitrary way (Brown motions).

Let us consider the **equation of motion**

$$\boldsymbol{F} = m\,\ddot{\boldsymbol{r}}, \tag{16}$$

where $\boldsymbol{F}$ - the known force, and m - mass of the material point. This force can be constant, i.e. independent neither on the position $\boldsymbol{r}$ nor time t. Integration (solution) of this equation gives the result

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\frac{\boldsymbol{F}}{m}t^2. \tag{17}$$

Initial conditions

If we assume that at the initial instant $t_0 = 0$ the material point had the position $\boldsymbol{r}_0$ and the initial velocity $\boldsymbol{v}_0$, its position can be defined at the every instant of time, as well as for $t > t_0$ or for $t < t_0$. One can predict future or tell what happened in the past. Such motion is described as reversible. The branch of physics dealing with the irreversible processes is called **thermodynamics.**

Reversibility

The problem of the reversibility in physics is one of the most important and not fully solved. Theories of the irreversible processes are very complicated and contain many difficult notions.

The very interesting situation is in the case when

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, t). \tag{18}$$

vector nonlinear function. The solutions of such equations very often depend very strongly on the initial-boundary conditions, what results in the so called deterministic chaos.

4.1 Newton second law of dynamics

Equation (16) defines the simplified version of the **Newton second law of dynamics**. It was assumed there that the force $\boldsymbol{F}$ and the mass m are independent on time t. Usually it is not a case.

In the situation that force and mass depend on time, we have:

$$\boldsymbol{F} = \boldsymbol{F}(t), \quad m = m(t). \tag{19}$$

Newton has formulated his second law in the following way:

$$\boldsymbol{F}(t) = \frac{d\,\boldsymbol{p}(t)}{d\,t}, \tag{20}$$

where $\boldsymbol{p} = m\,\boldsymbol{v}$ - momentum of a body. This equation can be written in the following form:

$$F(t) = \frac{d\,m(t)}{d\,t}\boldsymbol{v}(t) + m(t)\,\frac{d\,\boldsymbol{v}(t)}{d\,t}, \tag{21}$$

i.e. as compared with Eqn. (16) has one additional term. It is seen that in the Peyton's formulation if the force depends on time thanks to to changing in time mass and/or velocity.

Motion uniformly accelerated

The change of the velocity on the time unit is called the acceleration. If it is constant, then the **mean acceleration** $a_{śr}$ equals to the **momentary acceleration**:

$$a = a_{śr} = \frac{v - v_0}{t - t_0}, \tag{22}$$

where v - velocity at the moment t, v_0 - initial velocity at the moment t_0. If $t_0 = 0$,it follows from Eqn. (22) that:

$$v = v_0 + a\,t. \tag{23}$$

It is seen, that for $t = 0$ the velocity $v = v_0$, what agrees with the accepted assumption. Similarly, it was found that

$$x = x_0 + v_{śr}\,t, \tag{24}$$

where

$$v_{śr} = \frac{1}{2}(v_0 + v). \tag{25}$$

Putting Eqn. (25) into (24) and taking into account Eqn. (23) one obtains that

$$s = x = x_0 + v_0\,t + \frac{1}{2}a\,t^2. \tag{26}$$

Eqns. (23) and (26) are the two fundamental equations which allow to solve every problem of the uniformly accelerated motion. Let us note that they include 6 quantities: s - distance, x_0 - initial position, v_0 - initial velocity, v - momentary velocity, a - acceleration, t - time.

No force

In the case where no force is acting on a body the solution (17) of the equation of motion (16) takes a form

$$r = r_0 + v_0\, t. \tag{27}$$

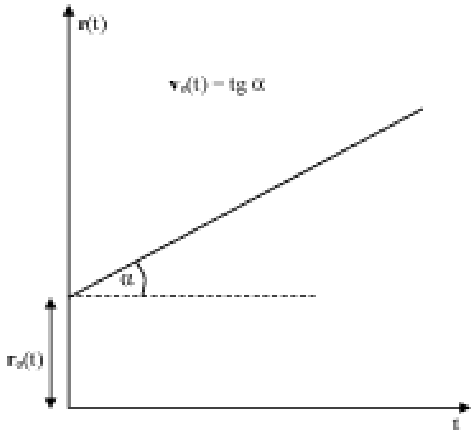

Figure 5: Motion of a material point when no force is acting

4.2 Constant force

The true dynamics starts at the moment when the sum of all acting forces is different from zero. Richness of diversity of phenomena is tremendous. We give some fundamental examples only.

4.2.1 The free drop of bodies

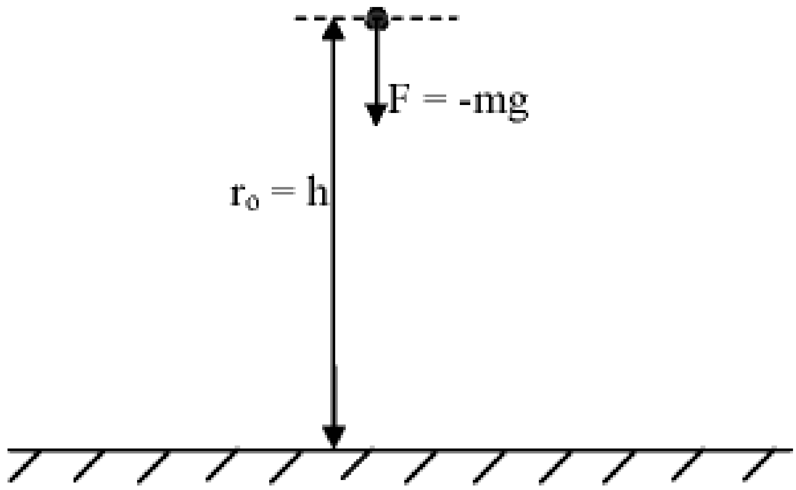

Figure 6: The free drop

Some questions appear: how long will be the body falling from a certain height h and what will be its velocity at the moment of hit at the surface?

At the instant of hit at the surface $r = 0$, so the solution (17) takes the form:

$$0 = h - \frac{m\,g}{2m} t^2, \tag{28}$$

and immediately

(29)

$$t = \pm\sqrt{\frac{2h}{g}}\,.$$

This result comes from mathematics. The answer on the second questions can be found from the mechanical energy conservation law: the potential energy of a body at the position $r = h$, when the body hits at the surface at the position $r = 0$ transforms totally in the kinetic energy, i.e.

$$m\,g\,h\; = \frac{1}{2}m\,v^2, \tag{30}$$

and finally

$$v = \pm\sqrt{2\,g\,h}\,. \tag{31}$$

Let us observe, that this result is in contradiction with our everyday observations. There is no mass m in Eqns. (29) and (31) . It follows that a feather and a very heavy weight fall at the same time and will have the same final velocity!

Today we know that the situation described above is an ideal one. On the Earth all bodies move not in a vacuum but in the air, and the forces of friction and the buoyancy slow down the motion. But the vacuum does not exist! So has been the opinion of Aristotle and all his followers for many centuries.

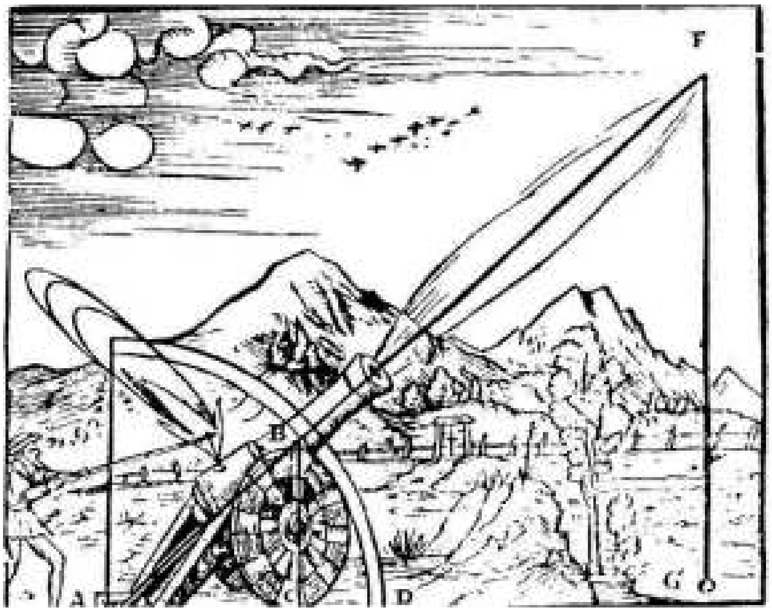

Figure 7: The throw (the shot of a canon) according to the physics: the ballistic can move along the lines only

4.2.2 Projectile motion

Let us consider the projectile motion of the material point with the initial velocity $v_0 = v(t)|_{t=0}$ and angle α_0 with the positive x - axis and the height h over the surface in the gravitational field with the acceleration g directed down to the surface (see Fig. 8). There is no resistance to the motion, there is no horizontal negative acceleration. The movement is flat: in the one vertical plane only (2D motion). The velocity $\boldsymbol{v}$ is always tangent to the trajectory: the trajectory changes its direction so the the velocity will change too.

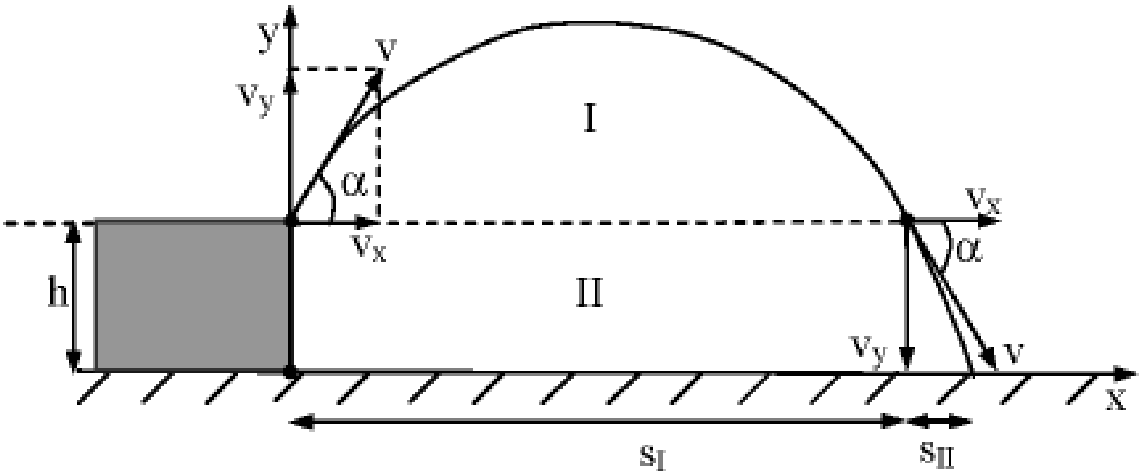

Figure 8: Projectile motion

It follows from Fig. 8 that at every instant

$$v_x = v\cos\alpha(t), \qquad v_y = v\sin\alpha(t). \tag{32}$$

In the moment when the trajectory reaches the level h again, the angle $\alpha^h = -\alpha^s$, so $v_y^h = -v_y^s$, where v_y^s - vertical initial velocity, a v_y^h - vertical velocity at the level h. At the point of the maximal height of the trajectory the angle $\alpha = 0^\circ$, thus $v_y = 0$.

The motion is studied in two regions: in the region I and in the region II.

The initial conditions in the region II are the final values in the region I.

The horizontal and vertical motions are independent.

The total covered distance equals $s = s_I + s_{II}$.

Motion in the region I

m - mass if the material point, α_0 – initial angle. Because $v_x = v_{0x} = v_0 \cos\alpha_0 = \text{const}$ (there is no horizontal acceleration) , we have

$$s_I = v_x\, t_I, \tag{33}$$

where t_I - time needed to reach the level h again. in the horizontal direction x the distance is given by the formula (24)

$$x = x_0 + v_{0x}\, t = v_{0x}\, t = v_0\, t\cos\alpha_0. \tag{34}$$

where $x_0 = 0$. time needed to cover the distance $x = s_I$

$$t = \frac{x}{v_0\cos\alpha_0}. \tag{35}$$

Vertical motion is the accelerated one, so

$$v_y(t) = v_0 \sin\alpha(0) - g\, t. \tag{36}$$

We know, that

$$y = h + v_{0y}\, t - \frac{1}{2} g\, t^2 = h + (v_0 \sin\alpha_0)t - \frac{1}{2} g\, t^2. \tag{37}$$

If we put into Eqn. (37) the time calculated from Eqn. (35), the equation for the trajectory takes the form $y = y(x)$

$$y = h + \sin\alpha_0 \frac{x}{\cos\alpha_0} - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\alpha_0} = h + x \operatorname{tg}\alpha_0 - \frac{g x^2}{2 v_0^2 \cos^2\alpha_0}. \tag{38}$$

This is the parabola equation with the arms directed to the down.

The distance in the x direction is calculated from the condition $y = h$, i.e.

$$x \left(\frac{g\,x}{2\,v_0^2 \cos^2\alpha_0} - \operatorname{tg}\alpha_0 \right) = 0. \tag{39}$$

Solution $x = 0$ is trivial and not interesting. The second solution has a form

$$x = \frac{2\,v_0^2}{g} \operatorname{tg}\alpha_0 \cos^2\alpha_0 = \frac{v_0^2}{g} \sin 2\alpha_0. \tag{40}$$

The maximal distance is calculated from the Eqn. (41) for the angles from the domain $\alpha_0 \in [0^\circ, 90^\circ]$, with the defined initial velocityv_0 and in the constant gravitational field g

$$\frac{d\,x}{d\,\alpha_0} = 2\,\frac{v_0^2}{g} \cos 2\alpha_0 = 0 \;\rightarrow\; \cos 2\alpha_0 = 0 \rightarrow \alpha_0 = 45^\circ. \tag{41}$$

Putting Eqn. (40) into Eqn. (35) the time for that motion can find

$$t = \frac{v_0^2 \sin 2\alpha_0}{v_0\,g\,\cos\alpha_0} = \frac{2}{g}\,v_0 \sin\alpha_0. \tag{42}$$

The biggest value is for the angle $\alpha_0 = 90^\circ$, i.e. for the vertical motion.

Motion in the region II

In the region II the drop and the simultaneous horizontal projection is observed.

in the starting point $v_{0x}^h = v_{0x}^s = \text{const}$, $v_{0y}^h = -v_{0y}^s$ (see Fig. 8). Vertical distance equals h, i.e.

$$h = -v_{0y}\,t - \frac{1}{2}\,g t^2. \tag{43}$$

It is a quadratic equation with respect to time $t = t_{II}$, so the solution has a form

$$t_{1,2} = \frac{-v_{0y} \pm \sqrt{v_{0y}^2 + 2\,g\,h}}{g}. \tag{44}$$

One of the solutions is always < 0 (for $h \geq 0$) (why?), and the second one is > 0.

The calculated distance x_{II} equals

$$x_{II} = v_{0x}\,t_2. \tag{45}$$

Summary

- Distance

(46)

$$s = x_I + x_{II} = \frac{v_0^2}{g}\sin 2\alpha_0 + v_{0x}\frac{-v_{0y} + \sqrt{v_{0y}^2 + 2\,g\,h}}{g}$$
$$= \frac{v_0}{g}\left(\frac{1}{2}\,v_0 \sin 2\alpha_0 + \cos\alpha_0\,\sqrt{v_0^2\sin^2\alpha_0 + 2\,g\,h}\right).$$

- Time

$$t = t_I + t_{II} = \frac{2}{g}\,v_0 \sin\alpha_0 + \frac{-v_{0y} + \sqrt{v_{0y}^2 + 2\,g\,h}}{g}$$
$$= \frac{1}{g}\left(v_0\sin\alpha_0 + \sqrt{v_0^2\sin^2\alpha_0 + 2\,g\,h}\right). \tag{47}$$

- Maximal ascend

Extremum of the quadratic equation:

$$y_{max} = \frac{v_0^2\sin^2\alpha_0}{2\,g} + h. \tag{48}$$

4.2.3 A body projected horizontally

A body is projected horizontally if angle $\alpha = 0$.

5 Friction

The friction is:

- static and kinetic, and
- gliding and rolling.

Sometimes it useless sometimes very needed. Have you some examples?

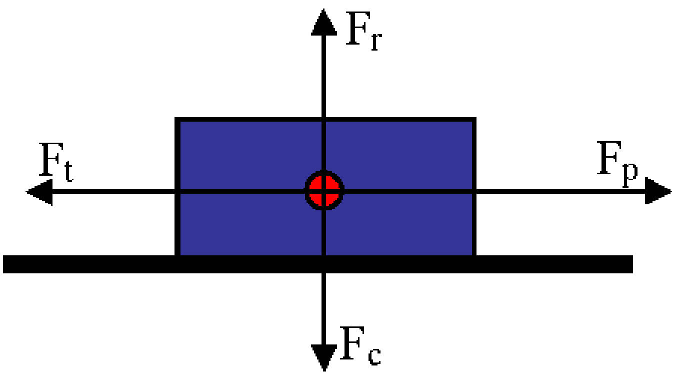

Figure 9: Gliding friction, $\boldsymbol{F}_c$ - gravity force, $\boldsymbol{F}_p$ - driving force, $\boldsymbol{F}_t$ - frictional force, $\boldsymbol{F}_r$ - reaction force

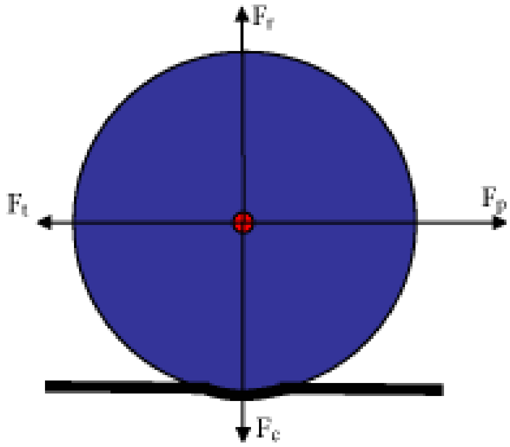

Figure 10: Rolling friction, $\boldsymbol{F}_c$ - gravity force, $\boldsymbol{F}_p$ - driving force, $\boldsymbol{F}_t$ - frictional force, $\boldsymbol{F}_r$ - reaction force

We assume in our model of friction that the friction force is proportional to force pressing the body to the ground

$$F_t = \mu \, F_c. \tag{49}$$

In Figs. 9 and 10 the pressing force is the same as the gravitational force. The only difference between the static and kinetic force is the friction coefficient μ.

Friction coefficients are dimensionless and their values are determined experimentally.

Friction in fluids is the more complicated phenomenon. It depends on the shape of the body and and on the character of the motion of the fluid (turbulent or laminar). Resistance force depends on the velocity as follows:

$$G = \frac{1}{2} C \, \rho \, S \, v^2, \tag{50}$$

where C - experimentally determined coefficient of the aerodynamic resistance, ρ - mass density of a fluid, S - crosssection of the body. Usually $0.4 < C < 1$.

Sometimes the collective coefficient is used

$$G = C_1 \, v^2. \tag{51}$$

The skiiers know the Eqn. (50) very well and they take the proper position during the skiing.

In the free drop if considering the air resistance we have

$$G - m \, g = m \, a, \tag{52}$$

where a - drop acceleration. There exists the certain limit (final) velocity.

In the limit case Eqn. (52) takes the form

$$G - m \, g = 0. \tag{53}$$

and after putting here Eqn. (50), we obtain

$$\frac{1}{2} C \, \rho \, S \, v^2 = m \, g, \tag{54}$$

and the final velocity v_g equals

$$v_g = \sqrt{\frac{2 \, m \, g}{C \, \rho \, S}}. \tag{55}$$

Table 1: Final velocities of some bodies in the air

Body	$v_g[m/s]$	$s_{95\%}[m]$
shot of the	145	2 500
a parachutist before the parachute is opened	60	430
a parachutist after the parachute is opened	5	3
tennis ball	31	115
basketball ball	20	47
ping-pong ball	9	10
rain drop ($\phi = 1.5$ mm)	7	6
your measurement	?	?

In the table the final velotities of v_g are given in [m/s] units and the distance $s_{95\%}$, which the body covers until the 95% of the final velocity is reached.

The End of the Lecture 4

6 Problems to think about

1. Answer the questions:

i. Give the characteristics of the rigid body. There exist in the nature the rigid bodies? Give some examples.

ii. How many degrees of freedom has the rigid body in the 3D space?

iii. How many degrees of freedom has the material point in the 3D space?

iv. What are the main differences between the material point and the rigid body?

v. What is the form of the second law of dynamics in the Newton's formulation? Give an example of the appliction of that formulation.

vi. What is the difference between the accelerated motion and the uniformly accelerated motion?

vii. What it is the ballistic curve?

2. Give the definition of the total energy of a body and their constituents. Some examples?

3. Give examples and explain why the friction is sometimes useful and sometimes harmful.