Physics for Computer Science Students

Lecture 3: Kinematics

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1 Frame of reference

Life: it is the motion. Everything what is alive befinds in the permanent motion. Everything around us and in us there is in a motion.

These sentences are true as far as something can be real. Let us notice the last phrase: "Everything around us..." That antropological point of view expresses the most important item in reserching the motion: the choice of the frame of reference.

Starting to study the motion we accept the seemingly obvious thrues about the existence of the matter and about the existence of the multidimensional space which is empty and without the structure.

(*In fact this is a philosophical question without the ultimate answer. We belive that the World exists objectively.*)

If one puts simply the material object in the space, it will be stated quite quickly, that there are no tools do describe the motion. In fact one can not judge if the the motion takes place at all.

This empty space is called the **physical space**. The following attributes can be assigned to the physical space:

- isotropy - no distinctive directions;
- homogeneity - no distinctive point.

Introduction into the space the material objects changes the stucture of the space. An examle of such a material object changing the structure of the space is e.g. the Earth. The presence of the Earth causes that certain directions start to be distinguished. E.g., if one wishes to throw a stone up or aside then one has to give the stone the starting velocity greater than zero. However, if one wishes to direct the stone down there is no need to give the stone the starting velocity: the stone falls down by itself.

In the empty space one has to give the stone **always** and **in the all directions** the non zero starting velocity. This kind of the space property is called the isotropy.

An example of the space atrtribute having the property of homogeneity is time. There are no the distinct instants.

(*This statement is not true to its end. Compare the discussion about the Big Bang in the book by M. Heller "The start is everywhere" (in Polish) or elsewhere*).

Any experiment can be conducted at any instants: the result will be always the same, if and only if the conditions in which the experiment was carried out were identical in all cases.

One can find the solution of the existence of the motion even in the ancient Greece. The famous Aristotle from Syracuse (287-212 BC) said: "Give me a point of support and I move the Earth."

(In the school handbooks this statement was cities by the occasion of the discussion of the lever (the simple machine), ut it can be interpreted in the more general manner.)

This point of support in the contemporary physics is called **the system of reference.**

Aristotle was waiting that somebody gives him this poit of support. But in vain. We are in the more comfortable situation, we can make use of his experience and take that "point of support" by ourselves.The comfort is the only thing which should be taken into accout by making choice of the system of reference.

If one can select one system of reference then the infinite numer of systems of reference can be selected as well. So, the obvious question arises: which one from the selected systems of reference is the best?

The answer is very simple: no one. All systems of reference are equivalent.

(*There is the group of the favourable systems of reference, called the inertial systems, i.e. those in which the I^{st} law of Newton is fulfilled, but it will be discussed later*.)

If one has to select a refernce system the best is of course that which was selected by us; we call that system **the absolute system of reference.** From now, all our discussions will be reffered to that abolute system of reference (e.g. in the case of the 2D space it can be the left upper corner of the computer screen).

Let us note, that **there exists an inseparable connection between mathematics and physics**. That close dependence stems from the way the physical theories are constructed. In any case, if one is going to discuss not only qualitative but alse the quantitative properties of the physical phenomenon one has to have the defined measures and ways how these measures can be applied and compared. The classical physics is based on the papers by **Isaac Newton** Philosophiae Naturalis Principia Mathematica, *Analysis Per Quanitatum Series Fluxiones, Ac Differentias: Cum Enumeratione Linearum Tertii Ordinis* and the papers by **Descartes Discours de la methode** (1637) .

Sir Isaac Newton (25th of December 1642 - 20th of March 1727, according to the Julian calendar, or 4th of January 1643 - 31st of March 1727, according to the Gregorian calendar) was the English physisist, mathematician, astronomer, Bible researcher, and alchemist. He was the first who showed that the same physical *laws rule the motions of the bodies on the Earth and in the Cosmos. He was the supporter of the heliocentric theory. He gave the mathematical justification of the Kepler's laws and showed additionaly that the orbits of plantes and comets can be eliptic and in the form of the other cone curves, i.e. hiperbolic or parbolic. He promoted the view, that the light consists of particles. He was the first who found out that the light decomposition into the separate colours made the prism is the property of the light and not of the prism (how it was formulated by Roger Bacon 400 hundreds years before). He formulated the law of the cooling down of the bodies, the conservation of momentum and the moment of momentum laws, he measured the velocity of sound in the air and formulated the theory of the origin of stars. He formulated the binomialtheorem and created the differential and variational calculus. He was the first who gave the mathematical description of the sea tides (1687).*

2 Co-ordinate systems

The very important turning point for our civilisation was the Cartesian algebraisation of the geometry. It became clear that to describe a point in the n-dimensional space it is enough to give the n numbers which can be interpreted as the coordinates of that point in the considered space. However, to be allowed to talk about the co-ordinates one has to define co-ordinate system. The most popular co-ordinate system is **the Cartesian co-ordinate system**. In the 3-dimensional (3D) space it is defined by the three lineary independent versors (base vectors) $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ (vectors of the unit lenght), which are pairwise perpendiculr, i.e. their scalar product vanishes

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad , \quad i,j = 1,2,3. \tag{1}$$

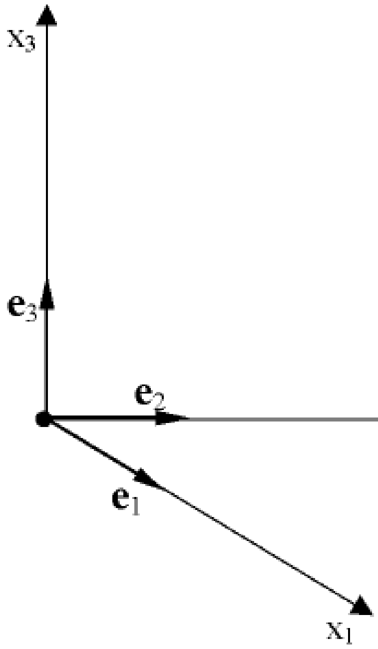

Fig. 1: Cartesian co-ordinate system

The simplests realisation of the base vector triad is to put them along the co-ordinate axies. The co-ordinate axies are most often denoted by the last letters of the alphabet, like x, y, z, and in the multidimensional spaces the indexing is used, like x_i, $i = 1, 2, ..., n$. So, in the three dimensional Euclid space the base vectors have the following co-ordinates (see Fig. 1):

$$\boldsymbol{e}_1 = (1, 0, 0), \boldsymbol{e}_2 = (0, 1, 0),\ \boldsymbol{e}_3 = (0, 0, 1)\ , \tag{2}$$

Linear independence of vectors:

every vector in the 3D space can be represented in the form of the linear combination of the three base vectors, i.e.

$$\boldsymbol{a} = a_1\,\boldsymbol{e}_1 + a_2\,\boldsymbol{e}_2 + a_3\,\boldsymbol{e}_3 = \sum_{i=1}^{3} a_i\,\boldsymbol{e}_i, \tag{3}$$

where the numbers $a_{1,}\,a_2, a_3$ are the not vanishing simultaneously the co-ordinates of the point pointed by the end of the vector $\boldsymbol{a}$.

*(It has a meanning if the vector **a** is a free vector, and its realisation has a begin at the begining of the co-ordinate system with the coordinates (0, 0, 0). The free vector it is a class of equivalence.)*

Eqn (3) can be written down as

$$\boldsymbol{a} = (a_1, a_2, a_3), \tag{4}$$

where the values of the coefficients in the sum (3). In the case when the vector $\boldsymbol{a}$ equals to one of the versors $\boldsymbol{e}_i$, Eqn (2) will has only one no vanishing coefficient $e_i = 1$.

Introduction of the base of the space with the help of the scalar product is very important because allows for the easy generalization of vector space base on the arbitrary space, e.g. the functional space.

The need of the introduction of such a functional space occures e.g. by the implementation of the Fourier fast transformation (FTT). We say that the functions $\phi_m(x)$, $m = 1, 2, ..., s$ generate the base of the functional space if their scalar product fulfils the following condition:

$$\int \phi_m \, \phi_n d\, x = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}, \quad m, n == 1, 2, ..., s. \tag{5}$$

The free vector defined by Eqn (3) (**mathematics!**), becomes the leading vector of the point M (**physics!**), if its begin will be attached to the begin of the co-ordinate system and the dependence on time will be introduced

$$M = M(x, y, z, t). \tag{6}$$

Here t i not a new co-ordinate but a certain parameter, so the following notation is more proper:

$$M = M(x(t), y(t), z(t)). \tag{7}$$

This seemingly a little important disinction has a very important e.g. in the Einstein's relativity theory. For any case the notation (7) will be used.

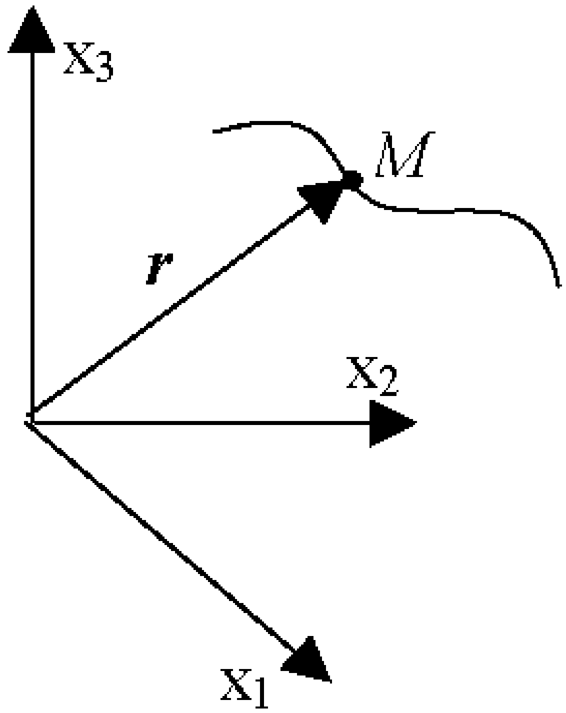

Fig. 2: The leading vector of the point M

The quatity $\frac{d\,\boldsymbol{r}}{d\,t}$ is called the velocity of the point M in the co-ordinate system x_1, x_2, x_3 iand is denoted by the letter $\boldsymbol{v}$

$$\begin{aligned}\boldsymbol{v} &= \frac{d\,\boldsymbol{r}}{d\,t} = \left(\frac{d\,x_1}{d\,t}, \frac{d\,x_2}{d\,t}, \frac{d\,x_3}{d\,t}\right) \\ &= \left(\dot{x}_1\boldsymbol{e}_1 + \dot{x}_2\boldsymbol{e}_2 + \dot{x}_3\boldsymbol{e}_3\right) = \sum_{i=1}^{3}\dot{x}_i\boldsymbol{e}_i,\end{aligned} \tag{8}$$

where the dot '.' over the symbol of the variable means its time derivative. The velocity of changing the velocity is called the acceleration $\boldsymbol{a}$

$$\boldsymbol{a} = \frac{d\,\boldsymbol{v}}{d\,t} = \frac{d^2\boldsymbol{r}}{d\,t} = \sum_{i=1}^{3}\ddot{x}_i\boldsymbol{e}_i. \tag{9}$$

Let us note that only the co-ordinates are differentiated - base vectors do not depend on time.

2.1 Polar co-ordinates

Let us consider the 2D space, and introduce there the polar co-ordinate system (r, ϕ), where r - the leading vector of the point M, and ϕ - the angle between the axis $0x_1$ of the rectangular co-ordinate system and the leading vector r, counted in the direction opposite to the motion of the clock hands.

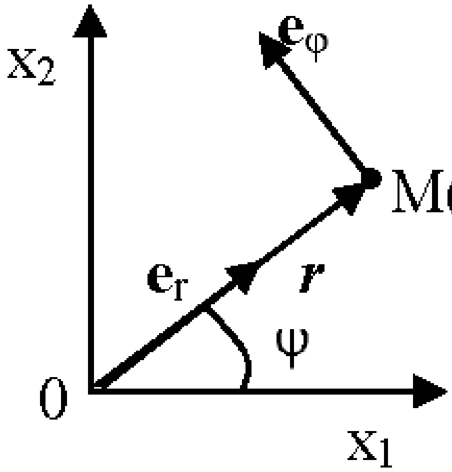

Fig. 3: Polar co-ordinate system

It can be stated easily that if the point M has in the rectangular co-ordinate system the co-ordinates (x_M,y_M), they can be expressed in the polar co-ordinate system as follows:

$$x_M = r_M \cos \phi_M, \quad y_M = r_M \sin \phi_M. \tag{10}$$

These expressions are unique with the accuracy to the multiply of the periods of the trigonometric functions, so the reciprocal dependencies can be found

$$r_M = \sqrt{x_M^2 + y_M^2}, \quad \phi_M = \arctan \frac{y_M}{x_M}. \tag{11}$$

The index M will be omitted if not needed. It follows fron Eqn (3) that

$$\boldsymbol{r} = a_r\, \boldsymbol{e}_r + a_\phi\, \boldsymbol{e}_\phi, \tag{12}$$

and it is obvious that $a_\phi = 0$, and $a_r = |\boldsymbol{r}| = r$. The position and orientation of the base vectors of the polar co-ordinate system is given in Fig. 3. Let us express them with the help of the vectors of the Cartesian co-ordinates. The construction in Fig. 4 can be helpful

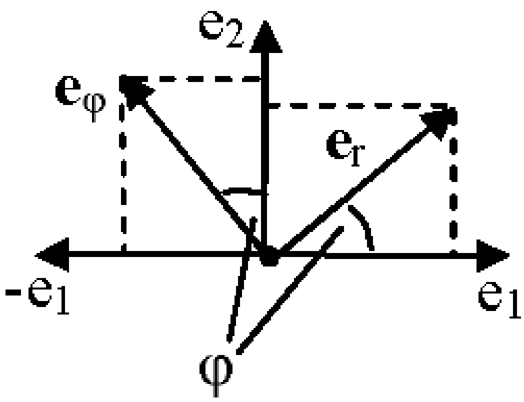

Fig. 4: Geometric construction of the polar co-ordinate system

It follows from the independence of the base vectors that the new base vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ are the linear combination of the old base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$\begin{aligned} \boldsymbol{e}_r &= a_1\,\boldsymbol{e}_1 + a_2\,\boldsymbol{e}_2, \\ \boldsymbol{e}_\phi &= a_3\,\boldsymbol{e}_1 + a_4\,\boldsymbol{e}_2, \end{aligned} \tag{13}$$

and from the condition that they have a unit length, i.e. that $e_\phi = e_r = 1$, it is obtained that

$$\begin{aligned} \boldsymbol{e}_r &= \cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2, \\ \boldsymbol{e}_\phi &= -\sin\phi\,\boldsymbol{e}_1 + \cos\phi\,\boldsymbol{e}_2. \end{aligned} \tag{14}$$

In the matrix notation Eqn (14) takes the form $\boldsymbol{e}^b = A\cdot\boldsymbol{e}^p$, or in more details

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix}, \tag{15}$$

where $\boldsymbol{e}^b = [\boldsymbol{e}_r, \boldsymbol{e}_\phi]^T$ - matrix of the base vectors of the polar co-ordinate system, A- transformation matrix of the co-ordinate systems, and $\boldsymbol{e}^p = [\boldsymbol{e}_1, \boldsymbol{e}_2]^T$- matrix of base vectors of the rectangular co-ordinate system. It is seen that the determinant of the transformation matrix $|A| = 1$.

If the angle ϕ depends on time, as it takes place in many cases, i.e.

$$\phi = \phi(t), \tag{16}$$

thus the base vectors depend of time too

$$\boldsymbol{e}_\phi = \boldsymbol{e}_\phi(t),\ \boldsymbol{e}_r = \boldsymbol{e}_r(t). \tag{17}$$

Let us calculate the velocity of the point M. It follows from Eqn (12) that

$$\boldsymbol{v}(t) = \frac{d\,\boldsymbol{r}}{d\,t} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\boldsymbol{e}}_r. \tag{18}$$

Making use of Eqn (13), it is obtained that

$$\dot{\boldsymbol{e}}_r = -\sin\phi\ \dot{\phi}\ \boldsymbol{e}_1 + \cos\phi\ \ \dot{\phi}\ \ \boldsymbol{e}_2 = \dot{\phi}\ \boldsymbol{e}_\phi. \tag{19}$$

It can be shown easily that

$$\dot{\boldsymbol{e}}_\phi = -\dot{\phi}\ \boldsymbol{e}_r. \tag{20}$$

Finally we have

$$\boldsymbol{v}(t) = \dot{r}\ \boldsymbol{e}_r + r\ \dot{\phi}\ \boldsymbol{e}_\phi. \tag{21}$$

Equation for the acceleration is a little bit more complicated

$$\boldsymbol{a}(t) = \frac{d\ \boldsymbol{v}}{d\ t} = \ddot{\boldsymbol{r}}\ \boldsymbol{e}_r + \dot{\boldsymbol{r}}\dot{\boldsymbol{e}}_r + \dot{\boldsymbol{r}}\ \dot{\phi}\,\boldsymbol{e}_\phi + \boldsymbol{r}\ \ddot{\phi}\,\boldsymbol{e}_\phi + \boldsymbol{r}\ \dot{\phi}\,\dot{\boldsymbol{e}}_\phi\,. \tag{22}$$

Making use of Eqns (19) and (20) one can obtain

$$\boldsymbol{a}(t) = (\ddot{r} - r\ \dot{\phi}^2)\boldsymbol{e}_r + (2\dot{r}\ \dot{\phi} + r\ \ddot{\phi})\ \boldsymbol{e}_\phi. \tag{23}$$

2.2 Normal co-ordinates

Normal co-ordinate system is used in physics quite often, e.g. in kinematics and acoustics. It is generated by the three versors

Normal co-ordinates are used in physics quite often, especially in kinematics and acoustics. It is created by three versors: $\boldsymbol{e}_\tau$ - tangent versor,

$\boldsymbol{e}_n$ - normal versor, and $\boldsymbol{e}_b$ - binormal versor.

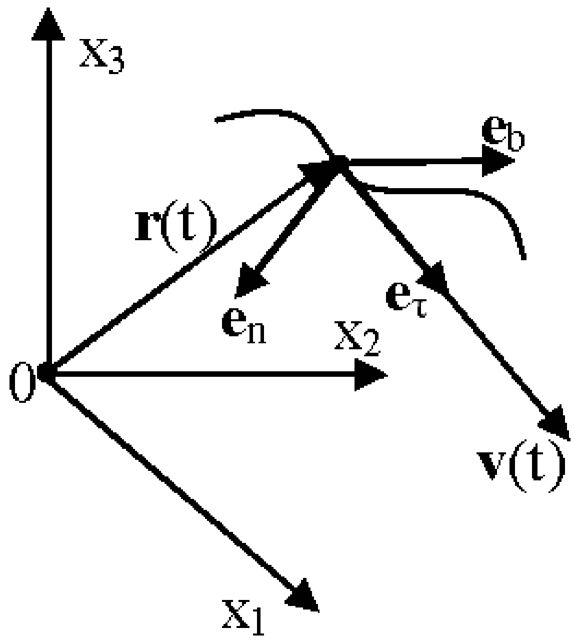

Fig. 5: Normal co-ordinates

The tangent versor $\boldsymbol{e}_\tau$ is tangent to the trajectory, i.e. it is parallel to the velocity vector $\boldsymbol{v}(t)$, i.e.

$$\boldsymbol{v}(t) = \frac{d\,\boldsymbol{r}}{d\,t} = v\,\boldsymbol{e}_\tau, \tag{24}$$

and the versor $\boldsymbol{e}_\tau$ can be calculated as follows

$$\boldsymbol{e}_\tau = \frac{\boldsymbol{v}(t)}{v} = \frac{\frac{d\,\boldsymbol{r}}{d\,t}}{\left|\frac{d\,\boldsymbol{r}}{d\,t}\right|}. \tag{25}$$

The normal versor $\boldsymbol{e}_n$ lyies on the plane perpendicular to the versor $\boldsymbol{e}_\tau$, so it has an infinite number of the possible orientations. There exists a simple procedure which allows to remove that ambiguity and to define it in a unique way. Let us notice that

$$\boldsymbol{e}_\tau \cdot \boldsymbol{e}_\tau = 1. \tag{26}$$

Let ud differentiate Eqn (26) with respect to time:

$$\frac{d\ \boldsymbol{e}_\tau}{d\ t}\boldsymbol{e}_\tau + \boldsymbol{e}_\tau\frac{d\ \boldsymbol{e}_\tau}{d\ t} = 0, \tag{27}$$

and it follows that $\boldsymbol{e}_\tau$ is perpendicular to $\frac{d\ \boldsymbol{e}_\tau}{d\ t}$. Thus one can define the versor $\boldsymbol{e}_n$ with the help of Eqn (28) in order to normalize it to the unit vector

$$\boldsymbol{e}_n = \frac{\frac{d\ \boldsymbol{e}_\tau}{d\ t}}{\left|\frac{d\ \boldsymbol{e}_\tau}{d\ t}\right|}. \tag{28}$$

In the 2D space the momentary curvature radius lyies alaways in the plane of the curve (because it is the only possibility) and is perpendiculat to the tangent versor $\boldsymbol{e}_\tau$. In the 3D space the problem is more difficult and the specific "social agreement" is needed. The "contract" says that direction of the momentary curvature radius of the trajectory (i.e. the normal direction) is parallel to the versor $\boldsymbol{e}_n$ given by Eqn (28).

The third versor of the triad versors of the normal co-ordinate system, the binormal versor is very easy to define. It is simply the result of the vector product of the tangent versor and the normal versor (for the right-handed system) - let us notice that the vector product is not alternable.

$$\boldsymbol{e}_b = \boldsymbol{e}_\tau \times \boldsymbol{e}_n. \tag{29}$$

The velocity vector is given by Eqn (24), so the accelaration can be find immediately

$$\boldsymbol{a} = \frac{d\ \boldsymbol{v}}{d\ t} = \dot{v}\ \boldsymbol{e}_\tau + v\ \ \dot{\boldsymbol{e}}_\tau = \dot{v}\ \boldsymbol{e}_\tau + \boldsymbol{v}\left|\frac{d\ \boldsymbol{e}_\tau}{d\ t}\right|\ \boldsymbol{e}_n. \tag{30}$$

It is seen that the acceleration has two components: the first one is tangent to the trajectory of the motion, and the second one describes the centripetal acceleration.

2.3 Cylindrical co-ordinates

The cylindrical co-oordinate system is generated by the three base versors: $\boldsymbol{e}_\rho$,$\boldsymbol{e}_\phi$ i $\boldsymbol{e}_z$ (see Fig. 6), where ρ - the radius of a cylinder on which our point lyies, ϕ - the angle between the x_1 axis and the projection of the leading vector $\boldsymbol{r}$ on the plane x_1x_2, and z - the distance x_3 over the plane x_1x_2. Simultaneously with the motion of a point pointed bythe leading vector $\boldsymbol{r}$, the base versors $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ change their orientation too.

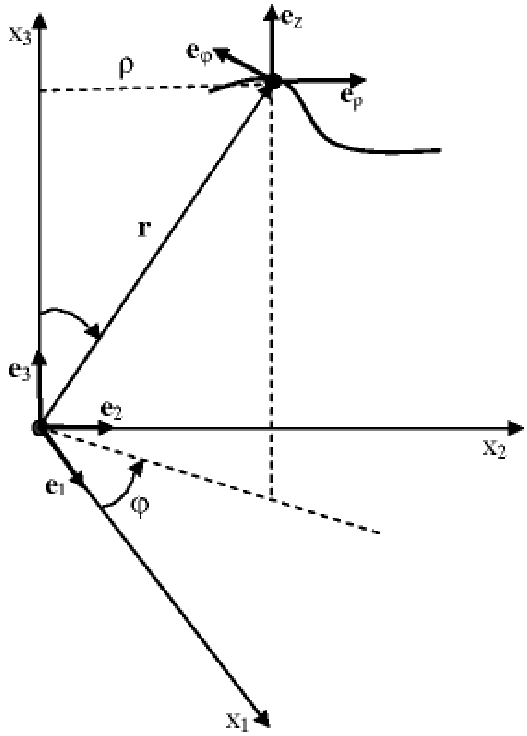

Fig. 6: Cylindrical co-ordinates

It is seen from Fig. 6 that the vector $\boldsymbol{r}$ has two components: one is parallel to the vector $\boldsymbol{e}_\rho$, and the second is parallel to the vector $\boldsymbol{e}_z$. It follows that

$$\boldsymbol{r}(t) = \rho\ \boldsymbol{e}_\rho + z\boldsymbol{e}_z. \tag{31}$$

and the velocity is easy to calculate

$$\boldsymbol{v}(t) = \frac{d\ \boldsymbol{r}}{d\ t} = \dot{\rho}\ \boldsymbol{e}_\rho + \rho\ \dot{\boldsymbol{e}}_\rho + \dot{z}\boldsymbol{e}_z. \tag{32}$$

In order to express that velocity as a vector in the cylindrical co-ordinate system, i.e. in the form $\boldsymbol{v} = v_\rho\boldsymbol{e}_\rho + v_\phi\boldsymbol{e}_\phi + v_z\boldsymbol{e}_z$,one has to express the versor $\boldsymbol{e}_\rho$ in the co-ordinates $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of the stationary co-ordinate system. It follows from Fig. 6 that

$$\boldsymbol{e}_\rho = \boldsymbol{e}_1 \cos\phi + \boldsymbol{e}_2 \sin\phi, \tag{33}$$

and that

$$\boldsymbol{e}_\phi = -\boldsymbol{e}_1 \sin\phi + \boldsymbol{e}_2 \cos\phi. \tag{34}$$

Because

$$\dot{\boldsymbol{e}}_\rho = -\boldsymbol{e}_1 \dot{\phi} \sin\phi + \boldsymbol{e}_2\, \dot{\phi} \cos\phi = \phi\, \boldsymbol{e}_\phi, \tag{35}$$

and

$$\dot{\boldsymbol{e}}_\phi = -\boldsymbol{e}_1 \dot{\phi} \cos\phi - \boldsymbol{e}_2\, \dot{\phi} \sin\phi = -\dot{\phi} \boldsymbol{e}_\rho, \tag{36}$$

the final form of the formula for the velocity in the cylindrical co-ordinates is as below

$$v(t) = \dot{\rho}\, \boldsymbol{e}_\rho + \rho\, \dot{\phi} \boldsymbol{e}_\phi + \dot{z}\, \boldsymbol{e}_z, \tag{37}$$

and the numerical value of the velocity equals

$$|v(t)| = \sqrt{(\dot{\rho})^2 + (\rho\, \dot{\phi})^2 + (\dot{z})^2}\;. \tag{38}$$

Now it is easy to find the vector of the acceleration

$$\begin{aligned} \boldsymbol{a}(t) &= \frac{d\ \boldsymbol{v}}{d\ t} \\ &= \ddot{\rho}\, \boldsymbol{e}_\rho + \dot{\rho}\, \dot{\boldsymbol{e}}_\rho + \dot{\rho}\, \dot{\phi}\, \boldsymbol{e}_\phi + \rho\, \ddot{\phi}\, \boldsymbol{e}_\phi + \rho\, \dot{\phi}\, \dot{\boldsymbol{e}}_\phi + \ddot{z}\, \boldsymbol{e}_z \\ &= (\ddot{\rho} - \rho(\dot{\phi})^2)\, \boldsymbol{e}_\rho + (2\dot{\rho}\, \dot{\phi} + \rho\, \ddot{\phi})\, \boldsymbol{e}_\phi + \ddot{z}\, \boldsymbol{e}_z, \end{aligned} \tag{39}$$

and its magnitude

$$|\boldsymbol{a}(t)| = \sqrt{(\ddot{\rho} - \rho(\dot{\phi})^2)^2 + (2\dot{\rho}\, \dot{\phi} + \rho\, \ddot{\phi})^2 + \ddot{z}^2}\,. \tag{40}$$

Example: Let us discuss the modeling of the uniform circular motion in the cylindrical co-ordinates. In that case z = const. (if $z \neq$ const., then one has to do with the elliptical or the spiral motion). So one can safely put

$$z = 0.$$

In the circular motion the radius is constant and one has obviously

$$\rho = R = \text{const.}$$

In the uniform motion the linear velocity is constant, i.e.

$$\phi = \omega\, t = \text{const.}$$

where ω - the constant (uniform) angle velocity. Making use of Eqns (31), (37) and (39) gives finally that

$$\begin{aligned} \boldsymbol{r}(t) &= R\, \boldsymbol{e}_\rho, \\ \boldsymbol{v}(t) &= \omega R\, \boldsymbol{e}_\phi, \\ \boldsymbol{a}(t) &= -\omega^2 R\, \boldsymbol{e}_\rho. \end{aligned} \tag{41}$$

i.e. exactly the description of the cirular motion.

2.4 Spherical co-ordinates

The spherical co-ordinate system is very useful too. It is generated by the three versors: $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$ (see Fig. 7).

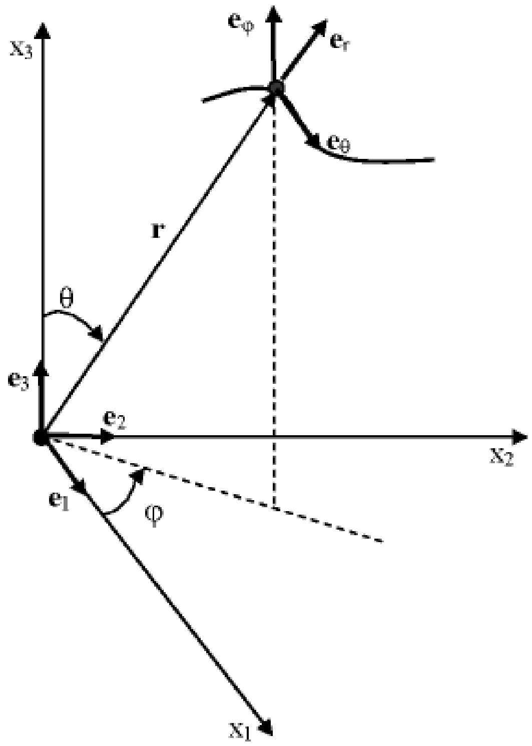

Fig. 7: The spherical co-ordinates

The connection of these new versors with the versors of the fixed Cartesian co-ordinate system is as follows:

$$\begin{aligned} \boldsymbol{e}_r &= \sin\theta(\cos\phi\ \boldsymbol{e}_1 + \sin\phi\ \boldsymbol{e}_2) + \cos\theta\ \boldsymbol{e}_3, \\ \boldsymbol{e}_\phi &= -\sin\phi\ \boldsymbol{e}_1 + \cos\phi\ \boldsymbol{e}_2, \\ \boldsymbol{e}_\theta &= \cos\theta(\cos\phi\ \boldsymbol{e}_1 + \sin\phi\ \boldsymbol{e}_2) - \sin\theta\, \boldsymbol{e}_3. \end{aligned} \tag{42}$$

The leading vector $\boldsymbol{r}(t)$ of the point in the spherical co-ordinate system is given be Eqn (41)

$$\boldsymbol{r}(t) = r\ \boldsymbol{e}_r. \tag{43}$$

Making use o the methodology used for the looking for the definitions of the physical quantities in the cylindrical co-ordinates it can be easily find that in the sperical co-ordinates

$$\boldsymbol{v}(t) = \frac{d\ \boldsymbol{r}}{d\ t} = \dot{r}\ \boldsymbol{e}_r + r\ \dot{\boldsymbol{e}}_r, \tag{44}$$

$$
\begin{aligned}
\dot{\boldsymbol{e}}_r &= \dot{\theta}\cos\theta(\cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2) + \sin\theta(-\dot{\phi}\sin\phi\,\boldsymbol{e}_1 + \dot{\phi}\cos\phi\,\boldsymbol{e}_2) - \dot{\theta} \\
&= \dot{\theta}\,\boldsymbol{e}_\theta + \dot{\phi}\sin\theta\,\boldsymbol{e}_\phi, \\
\dot{\boldsymbol{e}}_\phi &= -\dot{\phi}\cos\phi\,\boldsymbol{e}_1 - \dot{\phi}\sin\phi\,\boldsymbol{e}_2 = -\dot{\phi}\sin\theta\,\boldsymbol{e}_r - \dot{\phi}\cos\theta\,\boldsymbol{e}_\theta, \\
\dot{\boldsymbol{e}}_\theta &= -\dot{\theta}\sin\theta(\cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2) - \dot{\theta}\cos\phi\,\boldsymbol{e}_3 \\
&\quad + \cos\theta(-\dot{\phi}\sin\phi\,\boldsymbol{e}_1 + \dot{\phi}\cos\phi\,\boldsymbol{e}_2) \\
&= -\dot{\theta}\,\boldsymbol{e}_r + \dot{\phi}\cos\theta\,\boldsymbol{e}_\phi,
\end{aligned} \tag{45}
$$

$$
\boldsymbol{v}(t) = \dot{r}\,\boldsymbol{e}_r + r(\dot{\theta}\,\boldsymbol{e}_\theta + \dot{\phi}\sin\theta\,\boldsymbol{e}_\phi) \tag{46}
$$

$$
\begin{aligned}
\boldsymbol{a}(t) &= \frac{d\,\boldsymbol{v}}{d\,t} = \ddot{r}\,\boldsymbol{e}_r + \dot{r}\,\dot{\boldsymbol{e}}_r + r\,\dot{\phi}\dot{\theta}\cos\theta\,\boldsymbol{e}_\phi + r\,\ddot{\phi}\sin\theta\,\boldsymbol{e}_\phi \\
&\quad + r\,\dot{\phi}\sin\theta\,\dot{\boldsymbol{e}}_\phi + \dot{r}\,\dot{\phi}\sin\theta\,\boldsymbol{e}_\phi + \dot{r}\,\dot{\theta}\,\boldsymbol{e}_\theta + r\,\ddot{\theta}\,\boldsymbol{e}_\theta + r\,\dot{\theta}\,\dot{\boldsymbol{e}}_\theta.
\end{aligned} \tag{47}
$$

Making use of Eqns (42) and (45), Eqn (47) can be transformed and expressed in the co-ordinates with the versors $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$.

3 Kinematics of the material point

In physics, like in mathematics, the idealizations of certain objects and situations are used quite often. The idealized situation is the invaluable aid, because it allows to concentrate on the characteristic and the most important behaviours without going into the less important details.

An example of such an idealized object is the **material point**. It is taken from mathematics the 0-dimensional geometrical point with the additional attribute called **the mass**.

Mass is considered in physics in two contexts:

1. as the inertial mass,
2. as the heavy mass.

big should be a force to endow the body with the defined velocity: the force acting on the body with the greater mass should be greater.

The heavy mass manifets itself in the gravitational phenomena as a result of attraction of the material bodies. It is not obvious at all that these two kinds of masses (inertial and heavy) are equal to each other or not. Comparing the formulae for the force resulting from the Newton's second law and for the Newton's law of the universal gravity it can be state only that these two masses are proportional. Theirs equity can be proofed by the experiment only. Such an experiment was proposed by Newton. He took advantage of the fact that for small deflections of the pendulum, its period T equals

$$T = 2\pi\sqrt{\frac{m_b l}{m_g g}}, \tag{48}$$

where m_b - the inertial mass, m_g - the heavy mas, l - the length of the pendulum, g - acceleration of gravity. He made the experiments for the different substances and found that the period of the pendulum did not change. The conclusion was that $m_b / m_g = 1$, and the masses are equal.

The more precise experiments than Newton has done, were conducted by Roland Eötvös in years 1889-1908. Eötvös showed that in the limits of the experimental errors of the order $5 \cdot 10^{-9}$ both these masses inertial and heave are equal to each other. The precision of Eötvös was improved by several hundreds times in 1964 by R.H. Dicke, and the final result was the same - the both masses are equal. Reserching the cause of that equality led to the deeper understanding of the significance of the gravitation and to the formulation of the Einstein's general gravity theory.

The trace made by the material point during its motion is called **the trajectory of the material point.** This trajectory depends on the reference system.

In physics textbooks one can find several classical examples od such dependencies. Let us remind two of them.

- Material point freely droped down in the vehicle moving with the constant velocity in the reference system connected with that vehicle draws the trajectory in the form of the stright line directed downwards while in the reference system connected with the land the trajectory has the parabolic form.

- Material point put on the end of the propeller of the flying airplane in the refernce system connected with the airplane has a form of a circle while in the fixed reference system it has a form of a helix (spiral curve).

It is seen that the trajectory has a relative meanning. One can not talk about the trajectory in general. One has to talk about the trajectory in the well defined reference system. This fact is expressed in the context of the construction of the computer modelling of the physical phenomena.

3.1 Velocity

Kinematics studies the movement of the bodies without regarding the reasons triggered such their behaviour. The position of the body (e.g. the matrrial point or the extensive object) for the every instant of time *t*, is described by its **movement function** (trajectory)

$$\boldsymbol{x} = \boldsymbol{x}(t). \tag{49}$$

There are no constrains put on the form of the $\boldsymbol{x}$ function, but the one: this function has to be continous. From the pont of view of physics it means that it is impossble that at a certain instant the body dissapears in one place and suddenly appears in the another one. It means actually that theteleportation is not considered.

In the literature some first raports about the experimental confirmation of the teleportation on quantum level have appeared. Here the macro world is considered.

The trajectory $\boldsymbol{x}$ can have very simple form. It can be. e.g., the linear function of *t*

$$\boldsymbol{x} = \boldsymbol{A}\, t + \boldsymbol{B}. \tag{50}$$

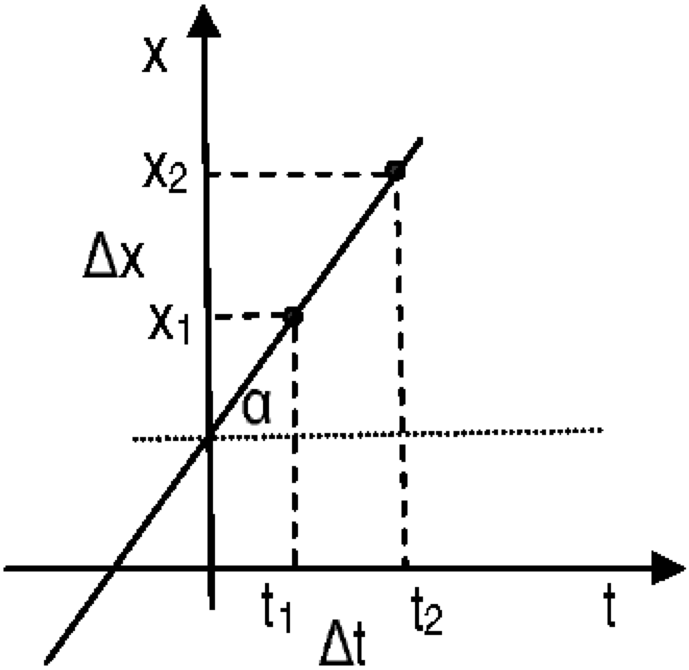

Fig. 8: Linear movement function

Fig. 8 shows the one-dimensional motion on the plane. In the case of the 3-D motion, as it is suggested by Eqn. (50), one has to do with the three directional angles α_i, $i = 1, 2, 3$, for the every axis separately. The angle α dictates the tempo of the changes of the covered distance: the greater α the smaller time $\Delta t = t_2 - t_1$ is needed to cover the defined distance $\Delta x = s = x_2 - x_1 = x(t_2) - x(t_1)$. It follows from Fig. 8 that

$$\frac{\Delta x}{\Delta t} = \tan \alpha. \tag{51}$$

This quantity is called the velocity and is denoted by the letter $v = |\boldsymbol{v}|$

$$\boldsymbol{v} = \frac{\Delta x}{\Delta t} = \frac{At_2 + B - At_1 - B}{t_1 - t_1} = A. \tag{52}$$

In the case described by Eqn. (50), the velocity v is constant for the every range Δt and for instants of time t, so it does not on t. Such a movent is called the **uniform motion** and is described by the equation

$$s = v\,t + s_0, \tag{53}$$

where s_0 - starting distance at the instant $t = t_0 = 0$.

The more complicated forms of the movement functions are the nonlinear functions. Here the huge possibilities stay for the disposal. The simplest nonliner functions the the polynomial functions, and the simplest of them is the trinomial square.

$$\boldsymbol{x}(t) = \boldsymbol{A}t^2 + \boldsymbol{B}\, t + \boldsymbol{C}. \tag{54}$$

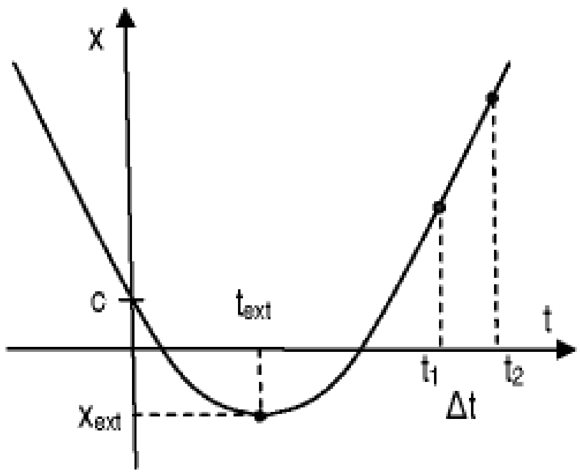

Fig. 9: The square function of movement

3.1.1 Characteristic points of the trinomial square

- The cross points of the curve with the 0t axis

$$t_1^0 = \frac{-B - \sqrt{\Delta}}{2A}, \quad t_2^0 = \frac{-B + \sqrt{\Delta}}{2A}, \quad \Delta = B^2 - 4\mathrm{AC}, \tag{55}$$

- The cross points of the curve with the 0x axis

$$t = 0, \quad x = C, \tag{56}$$

- The extremal point

$$t_{ext} = \frac{-B}{2A}, \quad x_{ext} = x(t_{ext}) = -\frac{B^2}{4A} + C. \tag{57}$$

If one tryies in a similar way to calculate the velocity in the case of the uniform motion then the certain difficulty appeas. In fact

$$\frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = A(t_2 + t_1) + B, \tag{58}$$

where B can be interpreted as the starting velocity. What is the meanning of Eqn. (58)? By the definition it is the **mean velocity** in the time interval $[t_1, t_2]$ for the movement function defined by Eqn. (54).

And what to do if one wants to know the **instant velocity** at an arbitrary instant of time t? Of course, one has to calculate the limit $t_2 \to t_1$. In this case $\Delta t \to 0$, and

$$v(t) = 2A\,t + B. \tag{59}$$

The same result can be obtaine more rapidly making use of the differential calculus. In this approach

$$v(t) = \frac{d\,x}{d\,t} = 2A\,t + B. \tag{60}$$

It follows from Eqn. (60) , that during the motion described by Eqn. (54) , the velocity depends on time t. Such movements are called **the variable motions**. In the first example given above the covered distance was the linear function of time and we called such a movement **the uniform motion**. Here the the velcity is a linear function of time, and such a movement we call **the uniformly variable motion**.

3.2 Acceleration

The rate of the velocity changes is called the acceleration and is usually denoted by the letter $\boldsymbol{a}$. It is a vector quantity and of course it depends on time, so $\boldsymbol{a} = \boldsymbol{a}(t)$, and

$$\boldsymbol{a}(t) = \frac{d\,\boldsymbol{v}(t)}{d\,t}. \tag{61}$$

In the case of the movement described by the movement function (54) the acceleration is constant and equals

$$\boldsymbol{a}(t) = 2\boldsymbol{A}. \tag{62}$$

It is the one additional reason why this movement is called **the uniformly variable motion**.

Let us note, that acceleration and velocity are the vector quantities. It means that they change if any of their attributes: direction, sense and absolute value (i.e. length), changes. Such situation is observed in the case of the uniform motion on a circle: the magniude of the velocity is constant while its direction is different at every point.

Summary

The definition of the reference system was introduced. Some examples of coordinate system were given. The three very important notions were introduced:

1. trajectory (or motion): $\boldsymbol{x} = \boldsymbol{x}(t)$;
2. velocity: $\boldsymbol{v} = \boldsymbol{v}(t) = \dfrac{d\,\boldsymbol{x}(t)}{d\,t}$;
3. acceleration: $a = a(t) = \dfrac{d\,\boldsymbol{v}(t)}{d\,t} = \dfrac{d^2\boldsymbol{x}(t)}{d\,t^2}$.

The End of the Lecture 3

4 Problems to think about

1. **Answer the questions:**

i. What is the difference between the reference system and the coordinate system?

ii. What should be the criteria in choice of the coordinate system to describe the given physical phenomenon?

iii. When the parabola arms are dircted down and when up? What is the geometrical interpretation of the trinomial square?

iv. Why there are two definitions of mass, what is their origin and what is their numerical value difference?

v. What it is the inertial reference system?

vi. What does it mean that the space is isotropic?

2. **Give the definitions of the trajectory, velocity and acceleration.**