Physics for Computer Science Students
Lecture 8

Mechanics of Liquids

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   - Equations of hydrodynamics for non-viscous liquids
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Definition of liquid

We have mentioned in Lecture 7 that the spherically-symmetric tension tensor had a form

$$S = \left\| \begin{array}{ccc} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{array} \right\| \tag{1}$$

i.e.

$$\begin{aligned} \sigma_x = \sigma_y = \sigma_z = -p\,, \\ \tau_x = \tau_y = \tau_z = 0\,. \end{aligned} \tag{2}$$

Materials with such properties are called the **non-viscous liquids**.

In non-viscous liquids exist normal tensions identical in all directions only and the tangential stresses vanish.

Definition of liquid

The vector of the tension tensor

$$\mathbf{S}_n = -p\,\mathbf{n}\,, \tag{3}$$

is always parallel to the normal vector $\mathbf{n}$ of the mentally separated surface, but with the opposite sign (this is our agreement).

The length of the tension vector

$$S_n = |\mathbf{S}_n| = \sqrt{S_{nx}^2 + S_{ny}^2 + S_{nz}^2} = p\,, \tag{4}$$

the value of the force acting on the surface element df equals $p\,df$.

Definition of liquid

Pascal law

It follows from Eqn (1) that the tensorial quadric is a sphere: this is exactly the Pascal law:
The pressure in a liquid propagates equally in all directions and always is directed perpendicularly to the border surface.

Isotropic body is described by the two Lamé material constants λ i μ in the general case. These constants define Lamé elastic potential

$$\mathcal{V} = -PJ_1 + \mu S_1 + \frac{\lambda}{2} J_1^2 \,, \tag{5}$$

P – (not)vanishing initial tension, $J_1 = \varepsilon_u + \varepsilon_v + \varepsilon_w$ – the first invariant,
$S_1 = \varepsilon_u^2 + \varepsilon_v^2 + \varepsilon_w^2$ – the symmetric second invariant.

Definition of liquid

Lamé potential

$$\begin{aligned} \sigma_i &= -P + \lambda J_1 + 2\mu\varepsilon_i \, , \\ \tau_i &= 2\mu\gamma_i \, , \qquad i = x, y, z \end{aligned} \tag{6}$$

Un the spherically-symmetric tension tensor the non-diagonal terms vanish, so $\mu = 0$, and liquids are characterized by the one elastic constant only, i.e by λ:

$$p = P - \lambda \, J_1 \, . \tag{7}$$

Conslusion: elastic energy is connected with the changing of the volume only. It follows that the non-viscous liquids do not resist to the changing of a shape: it can be accepted as a definition of a liquid.

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Equations of hydrodynamics for non-viscous liquids

Euler equations

Euler equations are used to calculate the distribution of the velocity in a liquid. stosujemy, gdy interesuje nas rozkład pola prędkości cieczy. Ogólne równanie ruchu mechaniki ośrodków ciągłych:

$$\rho \frac{d\mathbf{v}}{dt} = \rho\,\mathbf{F} + \mathrm{Div}\,S\,, \tag{8}$$

$$\mathrm{Div}\,S = -\,\mathrm{grad}\,p\,. \tag{9}$$

Continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + \mathrm{div}\,\rho\,\mathbf{v} = 0\,. \tag{10}$$

Equations of hydrodynamics for non-viscous liquids

Euler equations

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{F} - \mathrm{grad}\, p \, . \tag{11}$$

Thermodynamic state equation

$$f(\rho, p, T) = 0 \tag{12}$$

$\frac{d\mathbf{v}}{dt}$ – substantial derivative pochodna

We have five nonlinear equations (11), (10), (12) for five unknowns $\mathbf{v}$, ρ, and p. Solutions of these equations give the distributions of the velocity fields of the liquids and the trajectories of the particular elements of the liquids are unknown.

Equations of hydrodynamics for non-viscous liquids

Incompressible liquid

In the case of the incompressible liquid $\rho(x, t) = \text{const}$ two equations are sufficient:

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} &= \rho \mathbf{F} - \operatorname{grad} p \,, \\ \operatorname{div} \mathbf{v} &= 0 \,. \end{aligned} \tag{13}$$

Equations of hydrodynamics for non-viscous liquids

Lagrange equations

Lagrange equations are applied when the motion of the defined point (particle) of the liquid is the subject of our interest.
Let at the instant $t = 0$ it has the co-ordinates (a, b, c). We shall follow the motion of that point: $\mathbf{r} = \mathbf{r}(a, b, c, t)$. The velocity of that point is the function of time only and Eqn (11) taks the form

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = \mathbf{F} - \frac{1}{\rho}\,\mathrm{grad}\, p\,. \tag{14}$$

In Eqn 14 the co-ordinates (x, y, z) are present, while the independent co-ordinates are (a, b, c). After the change of the variables is made, the hydrodynamical Lagrange equations take the form:

Equations of hydrodynamics for non-viscous liquids

Lagrange equations

$$\mathcal{J}\left(\frac{\partial^2 \mathbf{r}}{\partial t^2} - \mathbf{F}\right) + \frac{1}{\rho}\,\mathrm{grad}_{(a,b,c)}\, p = 0\,, \tag{15}$$

$$\mathcal{J} = \left\|\begin{matrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{matrix}\right\| \tag{16}$$

Jacobian, i.e. the determinant of the matrix 17

$$\Delta = |\mathcal{J}| = \frac{\partial(x,y,z)}{\partial(a,b,c)}\,. \tag{17}$$

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Equations of hydrodynamics for viscous liquids

In real liquids the tangent stresses do not vanish, i.e. the friction between the neighbor layers of the liquid occurs. These layers define the direction of the motion of the liquid. Friction forces depend on the relative velocity of the layers

$$d\mathbf{v} = \dot{T}\, d\mathbf{r}\,. \tag{18}$$

$\dot{T}$ – tensor of the relative velocity.

Tensor $\dot{T}$ consists of two parts: $\dot{T} = \dot{T}^{(d)} + \dot{T}^{(a)}$, where $\dot{T}^{(s)} = \dot{T}^{(d)}$ – symmetric part of the tensor $\dot{T}$, and $\dot{T}^{(a)}$ – antisymmetric part of the tensor $\dot{T}$.

Equations of hydrodynamics for viscous liquids

Internal friction

Tensor $\dot{T}^{(a)}$ describes the rotational motion (of the rigid body type), so it does not influent on the internal friction. For the internal friction the tensor $\dot{T}^{(s)}$ is responsible:

$$\dot{T}^{(s)} = \left\| \begin{array}{ccc} \dot{\varepsilon}_x & \dot{\gamma}_z & \dot{\gamma}_y \\ \dot{\gamma}_z & \dot{\varepsilon}_y & \dot{\gamma}_x \\ \dot{\gamma}_y & \dot{\gamma}_x & \dot{\varepsilon}_z \end{array} \right\| , \tag{19}$$

$$\begin{aligned} &\dot{\varepsilon}_x = \frac{\partial v_x}{\partial x} , \qquad \dot{\varepsilon}_y = \frac{\partial v_y}{\partial y} , \qquad \dot{\varepsilon}_z = \frac{\partial v_x}{\partial z} , \\ &\dot{\gamma}_x = \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) , \quad \dot{\gamma}_y = \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) , \quad \dot{\gamma}_z = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) . \end{aligned} \tag{20}$$

Equations of hydrodynamics for viscous liquids

Internal friction

General equation of the continuous media mechanics

$$\rho \frac{d\mathbf{v}}{dt} = \rho\,\mathbf{F} + \mathrm{Div}(S^{(e)} + S^{(t)})\,, \tag{21}$$

Tensor S is divided into two parts: $S^{(e)}$ – elastic tensor, $S^{(t)}$ – tensor describing the internal friction and vanishing in the static case.

Let us consider a liquid flowing in the direction of the x axis with the velocity $\mathbf{u}$ growing in the y direction (Fig. 1). The force $\mathbf{i}\,\eta dx\,dz\,\partial u/\partial y$ acts on the particles under the element $df = dx\,dy$ and the force $-\mathbf{i}\,\eta\,dx\,dz\,\partial u/\partial y$ acts on the particles over the element $df = dx\,dy$. The force is proportional to the increase of the velocity and to the surface element,

η – the viscosity coefficient, or the coefficient of the internal friction.

Equations of hydrodynamics for viscous liquids

Internal friction

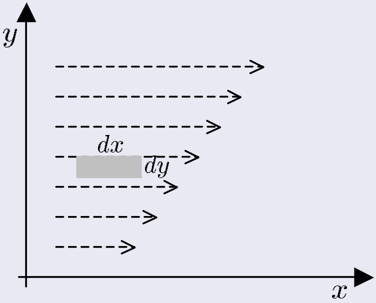

Fig. 1: Liquid layers and the surface element $df = dx\,dy$

Equations of hydrodynamics for viscous liquids

Navier-Stokes equation

$$\rho\frac{d\mathbf{v}}{dt} = \rho\,\mathbf{F} - \mathrm{grad}\,p + \frac{1}{3}\eta\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v} + \eta\triangle\mathbf{v}\,, \tag{22}$$

$\frac{d\mathbf{v}}{dt}$ – substantial derivative.

N-S equation is sometimes given in the form proposed by Maxwell

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} - \frac{1}{\rho}\,\mathrm{grad}\,p + \frac{1}{3}\varepsilon\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v} + \varepsilon\triangle\mathbf{v}\,, \tag{23}$$

ε – kinetic friction coefficient (or kinetic constant of viscosity).

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Formulation of a problem

Euler equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{F} - \mathrm{grad}\, p \,. \tag{24}$$

In the case of equilibrium $\mathbf{v} = 0$

$$\rho \mathbf{F} = \mathrm{grad}\, p \,. \tag{25}$$

This equation is valid both for the non-viscous and viscous liquids.

Hydrostatics

For incompressible liquids the force is the linear function of a potential. Let the field of the body forces comes from the potential of the gravitational forces

$$\mathbf{F} = -\mathbf{k}\, g = -\,\mathrm{grad}\, V\,. \tag{26}$$

$\mathbf{k}$ – unit vector in the direction of the z - axis, g – Earth acceleration.

$$V = g\, z + c \equiv g\, z\,. \tag{27}$$

Jeśli $V(z = 0) = 0$, to $c = 0$.

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Hydrostatics

Hydrostatic paradox

We have

$$p = -\rho g\ z\,. \tag{28}$$

If the liquids there is in the open vessel then on the surface of the liquid the pressure is constant and equal to the atmospheric pressure p_0. According to Eqn (28) surfaces of the equal pressure are horizontal so the surface of the liquid will be horizontal too.

Conclusion:

in the connected vessels the heights of the columns of liquid are equal and do not depend on the shape of the vessel – this phenomenon is called the hydrostatic paradox.

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Hydrostatics

Archimedes principle

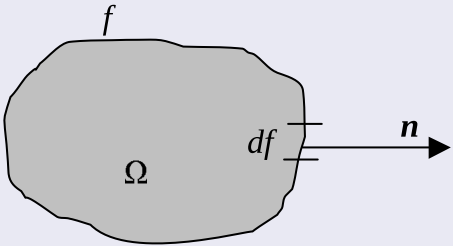

Fig. 2: Figure to derive the Archimedes principle

Hydrostatics

Archimedes principle

Let in a liquid in the equilibrium state with the acting body forces **F** only, there is a piece of matter having the volume Ω, and the area of the surface f (Fig. 2). Let us calculate the force exerted by the pressure of the liquid on the immersed body.

On the element df of the surface of a body acts the force $-\mathbf{n}\,p\,df$, i.e. the total force equals

$$\mathbf{P} = -\int_{\Omega} \operatorname{grad} p \, d\tau \,. \tag{29}$$

On the other hand

$$\operatorname{grad} p = \rho \, \mathbf{F} \,, \tag{30}$$

It follows from Eqn (29) and Eqn (30) $\rightsquigarrow$ that

$$\mathbf{P} = -\int_{\Omega} \rho \, \mathbf{F} \, d\tau \,. \tag{31}$$

Hydrostatics

Archimedes principle

In the case when $\mathbf{F}$ is the field of the gravitational forces, then $\mathbf{F} = -\mathbf{k}g$, and

$$\mathbf{P} = +g\mathbf{k} \int\limits_{\Omega} \rho \, d\tau \,. \tag{32}$$

Integral $m = \int_{\Omega} \rho \, d\tau$ – mass in the volume Ω, so $m_c = g \int_{\Omega} \rho \, d\tau$ – weight of the volume Ω. We have:
when a body is completely or partially immersed in a fluid (liquid, gas), the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.

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Bernoulli equation

Assumptions

1. liquid is non-viscous,
2. there is a functional relation $h = h(p)$,
3. motion is without rotation, so there exists such a potential φ, that $\mathbf{v} = -grad\varphi$,
4. body forces are potential too, so there exists such a V, that $\mathbf{F} = -\mathrm{grad}\, V$.

Caution: Bernoulli's principle applies only in certain situations!

Bernoulli equation

Euler equation according to assumptions 1. ad 2. can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \text{rot } v \times v = F - \mathrm{grad}(U + \frac{v^2}{2}), \tag{33}$$

$$U = \int_0^p \frac{dp}{\rho}. \tag{34}$$

Stokes law

Stokes law

The force acting on the sphere of the radius r and traveling with the velocity v in a liquid with viscosity coefficient η, but in such a manner the the relative motion of the liquid with respect to the sphere is laminar, equals

$$F = 6\pi\,\eta\;r\,v\,. \tag{35}$$

The force acting on the sphere placed in the flux of the liquid moving with the velocity v is the same.

Stokes law

Let the small ball with the radius r falls freely down in the viscous liquid. The gravitational force acts vertically down

$$F_d = \frac{4}{3}\pi\, r^3\, \rho\, g\,, \tag{36}$$

ρ – mass density of a ball.
in the up direction acts the buoyancy force

$$F_g = \frac{4}{3}\pi\, r^3\, \rho_c\, g\,, \tag{37}$$

ρ_c – mass density of a liquid.

Stokes law

At the beginning the ball moves with the certain acceleration, and with the increasing velocity the frictional force F increases too and after a certain time all forces are balanced

$$F_d - F_g - F = 0\,. \tag{38}$$

We have

$$\frac{4}{3}\pi\, r^3(\rho - \rho_c)\, g = 6\pi\,\eta\; r\; v\,, \tag{39}$$

and it follows that

$$v = \frac{2}{9}\,\frac{\rho - \rho_c}{\eta} g\, r^2\,. \tag{40}$$

Stokes law

Conclusions

- For small r the velocity of falling down is small, so eg. small drops of the rain or the smoke fall down very slowly.
- Stokes law can be applied to measure the viscosity of liquids and gases.

The End? :-(

The end of the lecture 8