Physics for Computer Science Students
Lecture 6

VIBRATIONS AND WAVES

Romuald Kotowski

Department of Applied Informatics

PJIIT 2009

Today: vibrations and waves!

Waves are everywhere and of every shape. We have surface and volume waves, we have sea and acoustic waves, we have ...
One have to distinguish waves and pulses:

- **Wave:** disturbance of a medium travelling with a defined velocity in the defined direction. In the case of the electromagnetic waves – it is the disturbace of the field.
- **Pulse:** measurable (changing in time) disturbance od the medium.

Fig. 1: The beachcomber (on the shallow water)

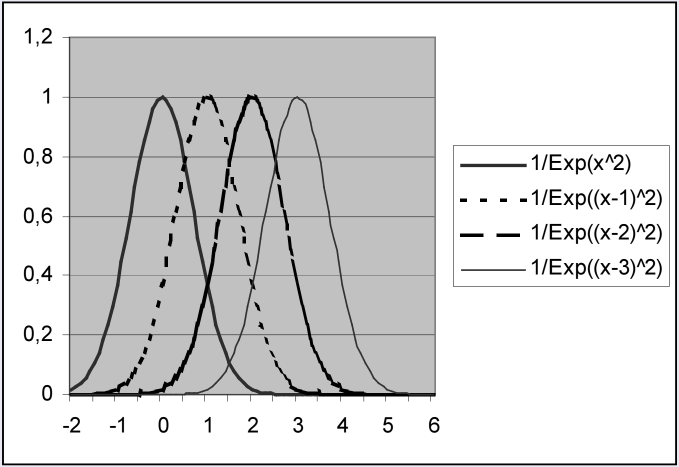

Fig. 2: Traveling pulse

Wave: arbitrary disturbance of a medium travelling with a defined velocity in the defined direction.

Table of contents

Waves in strings, long thin tubes, single way roads. . . i.e. waves propagating along defined lines.
One-dimensional wave is described by the function u of two co-ordinates – position x and time t: $\mathbf{u = u(x,t)}$

Kinematics and dynamics of waves is described by **PDE** (Partial Differential Equations) (systems of equations), because the function u depends on many variables.

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_{xt} = \frac{\partial^2 u}{\partial t\, \partial x}, \ldots$$

Below: some examples.

Example 1. Transport Equation

$$u_t + c\, u_x = 0\,.$$

describes e.g. polutant spilled into a fast moving stream.
$u(x,t)$ – the concentration of pollutant. Prior to the arrival of the pollutant at the position x, the value $u = 0$.

Example 2. Diffusion equation, conductivity equation (heat, electric current)

$$u_t = D\, u_{xx}\,.$$

Example 3. Linearized Burgers equation

$$u_t + c\, u_x = D\, u_{xx}\,,$$

a combination of the transport and diffusion processes.

Example 4. Nonlinear Burgers equation

$$u_t + u\, u_x = D\, u_{xx}\,,$$

a fundamental equation from fluid mechanics that combines a different advection processes with diffusion. For $D = 0$ it becomes the **inviscid Burgers equation**

$$u_t + u\, u_x = 0\,,$$

classical example of shock waves.

Example 5. Equation of vibrating string – wave equation

$$u_{tt} = c^2 \, u_{xx} \, .$$

it not suggest that this is the only equation which describes wave behavior.

Example 6. Korteweg-deVries equation

$$u_t + u \, u_x + u_{xxx} = 0 \, ,$$

was derived in 1895 by Korteweg and deVriesa to model waves on the surface of relatively shallow water. Of particular interest are solutions of this equations called *solitary* waves or *solitons*.

Table of contents

Traveling wave

$$u(x,t) = f(x - c\,t)\,, \tag{1}$$

f – function of one variable, c – constant $\neq 0$.
If $c > 0$ – wave travels with the velocity c in the positive sense of the direction of the co-ordinate axis; for $c < 0$ – in opposite.

Example 7. Find the solution of the wave equation

$$u_{tt} = a\,u_{xx}\,, \quad \text{constant } a > 0\,,$$

in the form of the travelling wave.

We assume the solution in the form $u(x,t) = f(x - c\,t)$ and differentiate

$$u_t(x,t) = [f'(x - c\,t)](x - c\,t)_t = -c\,f'(x - c\,t)\,,$$
$$u_x(x,t) = [f'(x - c\,t)](x - c\,t)_x = f'(x - c\,t)\,.$$

and once again

$$u_{tt}(x,t) = [-cf''(x - c\,t)](x - c\,t)_t = c^2\,f''(x - c\,t)\,,$$
$$u_{xx}(x,t) = [-cf''(x - c\,t)](x - c\,t)_x = f''(x - c\,t)\,.$$

We put the obtained result into the wave equation

$$c^2\, f''(x - c\, t) = a\, f''(x - c\, t)\,.$$

Putting $z = (x - c\, t)$ we obtain

$$(c^2 - a)\, f''(z) = 0\,,$$

for all z .

- if $c^2 = a$

$$u(x,t) = f\,(x - \sqrt{a}\, t)\,, \quad u(x,t) = f\,(x + \sqrt{a}\, t)\,.$$

Examples of solutions:

$$u(x,t) = \sin(x - \sqrt{a}\, t)\,,$$

$$u(x,t) = (x + \sqrt{a}\, t)^4\,,$$

$$u(x,t) = e^{-(x - \sqrt{a}\, t)^2}\,;$$

- if $f'' = 0$

$$f(z) = A + B\,z\,,$$

$B \neq 0$ in order the profil f is not constant.

Table of contents

Wave front – e.g. a sudden change in weather (see Fig. 3.

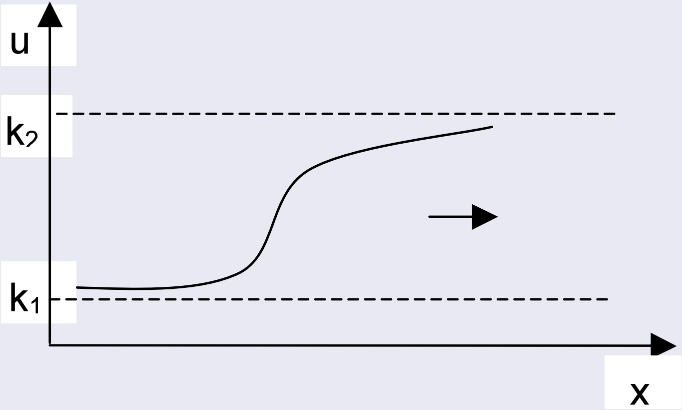

Fig. 3: Profil fali w chwili t

Traveling wave $u(x,t)$ is **the wave front**, if for an arbitrary instant of time t

$$u(x,t) \longrightarrow k_1\,,\ \text{gdy}\ x \longrightarrow -\infty\,,$$

$$u(x,t) \longrightarrow k_2\,,\ \text{gdy}\ x \longrightarrow \infty\,,$$

for certain constants k_1 i k_2 .
In the case when $k_1 = k_2$ the wave front is called **the pulse**.

Table of contents

The traveling wave of the type $u(x,t) = \cos(2x + 6t)$ is neither a wave front nor a pulse – it is ana example of the another type of a wave.

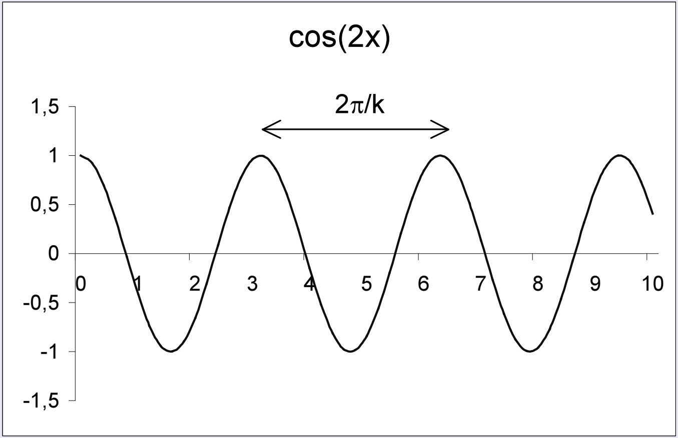

Fig. 4: One cycle of a wave train

A traveling wave which can be written in the form

$$u(x,t) = A\cos(kx - \omega t) \quad \text{lub} \quad u(x,t) = A\cos(kx + \omega t)\,,$$

$A \neq 0\,,\ k > 0\ $ i $\ \omega > 0$ – constants, is called **a wave train**.
After rewriting

$$u(x,t) = A\cos\left[k\left(x - \frac{\omega}{k}\,t\right)\right]\,,$$

it is seen that it is

- the traveling wave $u(x,t) = f(kx - \omega t)$;
- with a profile $f(z) = A \cos(kz)$;
- traveling with a velocity $c = \omega / k$ (see Fig. 4);
- $f(z)$ is a periodic function.

k – **wave number**, giving a number of cycles in the window of the length $= 2\pi$;
ω – **circular frequency**, defines number of wave cycles at the point x in the time interval 2π.

Not all k i ω are permitted. Relation between ω and k is called the dispersion equation

$$\omega = \omega(k)\,,$$

Example 8. Klein-Gordon equation

$$u_{tt} = a\,u_{xx} - b\,u\,, \tag{2}$$

$a\,,\ b$ – constants, $>0\,,$
models the transverse vibration of a string with a linear restoring force.

The wave train is a solution of this equation if

$$-\omega^2 A \cos(kx - \omega t) = a[-k^2 A \cos(kx - \omega t)] - b A \cos(kx - \omega t)\,, \tag{3}$$

or

$$A(\omega^2 - ak^2 - b) \cos(kx - \omega t) = 0\,. \tag{4}$$

Dispersion equation $\omega^2 = ak^2 + b$, i.e. $\omega = \sqrt{ak^2 + b}$, and thus

$$u(x,t) = A\cos\left(kx - \sqrt{ak^2+b}\;t\right) = A\left[k\left(x - \sqrt{\frac{ak^2+b}{k^2}}\;t\right)\right], \tag{5}$$

travels with the velocity

$$c = \sqrt{\frac{ak^2+b}{k^2}} = \sqrt{a + \frac{b}{k^2}} = \sqrt{a + \frac{ab}{\omega^2 - b}}, \tag{6}$$

wave train with the greater frequency travel with a smaller speed.
Klein-Gordon equation is dispersive.

Example 9. **Transport equation**

$$u_t + a\,u_x = 0\,. \tag{7}$$

Wave train is a solution if

$$\omega\,A\,\sin(kx-\omega t) + a[-kA\,\sin(kx-\omega t)] = 0\,, \tag{8}$$

or

$$A(\omega - a\,k)\sin(kx-\omega t) = 0\,, \tag{9}$$

dispersion $\omega = a\,k$.
For every wave number wave train travels with the constant velocity $c = a$. **Transport equation is not dispersive**.

Solitons

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

J.S. Russel, 1844

Solitons

Throughout his life Russell remained convinced that his solitary wave (the *Wave of Translation*) was of fundamental importance, but nineteenth and early twentieth century scientists thought otherwise. His fame has rested on other achievements. To mention some of his many and varied activities, he developed the "wave line" system of hull construction which revolutionized nineteenth century naval architecture, and was awarded the gold medal of the Royal Society of Edinburgh in 1837. He began steam carriage service between Glasgow and Paisley in 1834, and made one of the first experimental observations of the "Doppler shift" of sound frequency as a train passes. He reorganized the Royal Society of Arts, founded the Institution of Naval Architects and in 1849 was elected Fellow of the Royal Society of London. He designed (with Brunel) the "Great Eastern" and built it; he designed the Vienna Rotunda and helped to design Britain's first armored warship (the "Warrior"). He developed a curriculum for technical education in Britain, and it has recently become known that he attempted to negotiate peace during the American Civil War.

Solitons

In 1895 Korteweg and de Vries obtained an equation modeling the height of the surface of the shallow water in the presence of long wave gravitational waves. For such waves the wavelength is big as compared with the depth of the water.

$$U_t + (a_1 + a_2 U) U_x + a_3 U_{xxx} = 0\,, \quad a_2,\ a_3 \neq 0\,. \tag{10}$$

It is the third order nonlinear differential equation. jest The replacement $u = a_1 + a_2 U$ and re-scaling of the independent variables x and t gives

$$u_t + u\, u_x + u_{xxx} = 0\,. \tag{11}$$

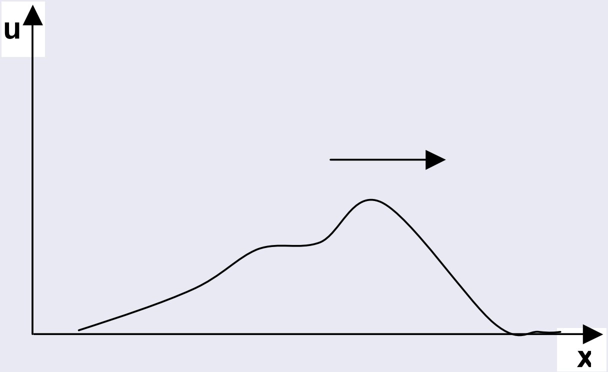

Fig. 5: Pulsu profile for which $u(x,t)$, $u_x(x,t)$ and $u_{xxx}(x,t) \longrightarrow 0$, when $x \longrightarrow \pm\infty$.

We are looking for a solution $u(x,t) = f(x - ct)$ in a form of a pulse with $c > 0$ and with $u(x,t)$, $u_x(x,t)$ and $u_{xxx}(x,t) \to 0$, when $x \to \pm\infty$ (see Fig. 5). We obtain

$$-cf' + ff' + f''' = 0\,. \tag{12}$$

Integrate once again

$$-cf + \frac{1}{2}f^2 + f'' = a\,, \tag{13}$$

a – integration constant.
Because $f(z)$ and $f''(z)$ for $z \to \pm\infty$, a has to vanish.

Multiply by f' and integrate once again

$$-\frac{1}{2}cf^2 + \frac{1}{6}f^3 + \frac{1}{2}(f')^2 = b\,. \tag{14}$$

By the condition of vanishing in infinity $b = 0$.
We solve with respect to $(f')^2$

$$3(f')^2 = (3c - f)f^2\,. \tag{15}$$

We substitute: $g^2 = 3c - f$; and it follows: $f = 3c - g^2$, $f' = -2gg'$.

$$\frac{2\sqrt{3}}{3c - g^2}\; g' = -1\,. \tag{16}$$

We decompose into simple fractions and integrate with respect to z:

$$\ln\left(\frac{\sqrt{3c}+g}{\sqrt{3c}-g}\right) = -\sqrt{c}\,z + d\,, \qquad (17)$$

d – integration constant. We solve with respect to g

$$g(z) = \sqrt{3c}\,\frac{\exp(-\sqrt{c}\,z+d)-1}{\exp(-\sqrt{c}\,z+d)+1} = -\sqrt{3c}\,\mathrm{tgh}\left[\frac{1}{2}(\sqrt{c}\,z-d)\right]\,, \qquad (18)$$

In the old notation

$$f(z) = 3c\ \operatorname{sech}^2[\frac{1}{2}(\sqrt{c}\,z - d)]\,. \qquad (19)$$

Recalling :

$$\operatorname{sech}(z) = 1/\cosh(z)\,, \quad \cosh(z) = \frac{1}{2}(e^z + e^{-z})\,.$$

d does not influence the solution (the argument is shifted only) argumentu), so we put $d = 0$,

$$\boxed{u(x,t) = 3c\,\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct)\right]\,.}$$

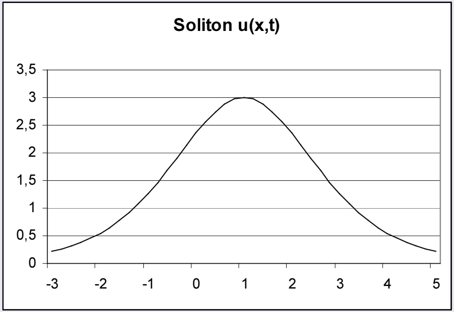

Fig. 6: Profile of the solution of the KdV equation (soliton)

Russel has found, that his waves on the water travel faster if they are higher. Our solution confirms his observation, because the amplitude is proportional to c ($= 3c$.

Fig. 7: The Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995. For the technically minded, the aqueduct is 89.3 m long, 4.13m wide, and 1.52m deep.

Fig. 8: Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995

The 'Wave of Translation' itself was regarded as a curiosity until the 1960s when scientists began to use modern digital computers to study non-linear wave propagation. Then an explosion of activity occurred when it was discovered that many phenomena in physics, electronics and biology can be described by the mathematical and physical theory of the 'soliton', as Scott Russell's wave is now known. This work has continued and currently includes modeling high temperature superconductors and energy transport in DNA, as well as in the development of new mathematical techniques and concepts underpinning further developments.
After a delay which would probably be unacceptable to present day funding bodies, and in a field he could never have dreamed of, Scott Russell's observations and research of 160 years ago have hit the big time in the present day fibre-optic communications industry. The qualities of the soliton wave which excited him (the fact that it does not break up, spread out or lose strength over distance) make it ideal for fibre-optic communications networks where billions of solitons per second carry information down fibre circuits for cable TV, telephone and computers ("The secrets of everlasting life", New Scientist 15 April 1995). It is fitting that a fibre-optic cable linking Edinburgh and Glasgow now runs beneath the very tow-path from which John Scott Russell made his initial observations, and along the aqueduct which now bears his name.

The wave equation

$$u_{tt} = c^2\, u_{xx}\ , \tag{20}$$

models the vibration of the tensed string (e.g. in the guitar).

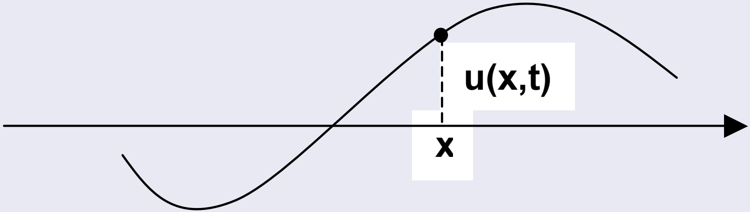

Fig. 9: Displacement $u(x,t)$ at the instant t in the position x

$u(x,t)$ – measure of the displacement of the string in the position x at the instant t; $u_t(x,t)$ – vertical velocity of the point x on the string at the instant t; $u_{tt}(x,t)$ – vertical acceleration of the point x on the string at the instant t; $u_x(x,t)$ – measure of the inclination of the string in the position x.

Vibrations depend on the material of the string and on the value of the force tensing the string. We make the following assumptions:

- the string is uniform: the density of mass ρ on the unit of length is constant;
- vibrations are flat: string remains in its plane of vibrations;
- the tension is uniform: every part of the string acts on neighbors with the same force T; direction of the force changes, it is always tangent to the string zmienia;
- no other forces;
- small vibrations: slope u_x is always small.

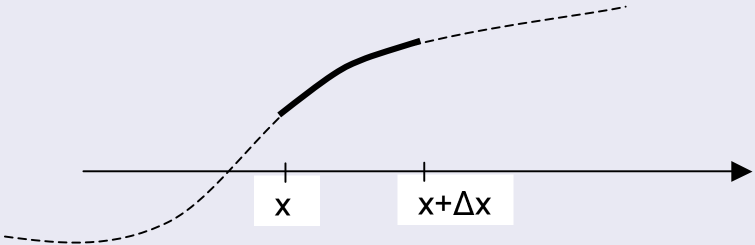

Fig. 10: Part of a string S

Let S be a segment between points x and $x + \Delta x$, where $\Delta x > 0$ is small (Fig. 10). Wave equation is a conclusion from the second law of Newton, which says that

$$(\text{Masa } S)\cdot (\text{Acceleration } S) = \text{Total force acting on } S\,, \tag{21}$$

where acceleration and force act perpendicularly to S.

Mass of the string segment S:

$$\text{Masa } S = \rho \int_{x}^{x+\Delta x} \sqrt{1 + (u_x(s,t))^2}\, ds\,. \tag{22}$$

for small amplitude $|u_x| \ll 1$, więc

$$\text{Mass } S = \rho \int_{x}^{x+\Delta x} 1\, ds = \rho\, \Delta x\,. \tag{23}$$

Acceleration of the string segment S:

$$u_{tt}(x,t) \tag{24}$$

Force acting on a string segment S: tangent vector to the string at the point x has the co-ordinates $-(1, u_x(x,t))$, so the stretching force T acting on the left end of a segment:

$$-T\,\frac{(1, u_x(x,t))}{\sqrt{1+(u_x(s,t))^2}}\,, \tag{25}$$

Making use of the assumption of the small amplitudes once again

$$\sqrt{1+(u_x(s,t))^2} \approx 1\,,$$

vertical component of the force equals

$$-T\,u_x(x,t)\,.$$

We repeat the considerations for right end of the segment

$$T\, u_x(x + \Delta x, t)\,.$$

so, the total force F_c acting on S equals

$$F_c = T\, u_x(x + \Delta x, t) - T\, u_x(x, t)\,. \tag{26}$$

The obtained results we put into Eqn. (21)

$$(\rho \, \Delta x) \, u_{tt}(x,t) = T \, u_x(x+\Delta x, t) - T \, u_x(x,t) \, . \tag{27}$$

divide by Δx

$$\rho \, u_{tt}(x,t) = T \, \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} \, ,$$

and in the limit $\Delta x \to 0$ gives

$$\rho \, u_{tt}(x,t) = T \, u_{xx}(x,t) \, .$$

Putting $c = \sqrt{T/\rho}$ we obtain the standard form of the wave equation

$$\boxed{u_{tt}(x,t) = c^2\, u_{xx}(x,t)\,.} \tag{28}$$

The equation is more complicated when the other forces are included, e.g.

$$\rho\, u_{tt} = T\, u_{xx} - F\, u_t - R\, u + f(x,t)\,.$$

$-F\, u_t$ – friction force (const. $= F > 0$);
$-R\, u$ – linear back force (const. $= R > 0$);
$+f(x,t)$ – external force(e.g. gravitation).

Solutions are the traveling waves

$$u(x,t) = f(x - c\,t)\,, \qquad u(x,t) = f(x + c\,t)\,,$$

where c is the propagation velocity of the wave. Because

$$c = \sqrt{T/\rho}\,,$$

the velocity of the wave can be:

- growing, when the string tension T will be growing,
- reducing, by taking the material with the greater mass density.

We show that the solution of the wave equation $u_{tt} = c^2 u_{xx}$ is the sum of the two solutions: one traveling to right and the second traveling to the left

$$u(x,t) = F(x - ct) + G(x + ct)\,.$$

Initial condition:

- partial differential equation

$$u_{tt} = c^2 u_{xx}\,, \qquad -\infty < x < \infty\,,\; t > 0\,,$$

- initial conditions

$$u(x,0) = f(x)\,,$$
$$u_t(x,0) = g(x)\,,$$

can be formulated as follows:

$$u(x,t) = \frac{1}{2}(f(x - ct) + f(x + ct)) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\,.$$

We know that the solutions are the two traveling waves: $h(x-cy)$ and $h(x+ct)$. We make the change of the variables:

$$\xi(x,t) = x - ct\,, \qquad \eta(x,t) = x + ct\,,$$

these are the co-ordinates 'following' the waves traveling from the left and from the right. The construction of the solution is easer. From the definition

$$u(x,t) = U(\xi(x,t), \eta(x,t))\,.$$

We differentiate

$$\begin{aligned}
u_t &= U_\xi \xi_t + U_\eta \eta_t = -cU_\xi + cU_\eta \, , \\
u_{tt} &= -c(U_{\xi\xi}\xi_t + U_{\xi\eta}\eta_t) + c(U_{\eta\xi}\xi_t + U_{\eta\eta}\eta_t) \\
&= -c(-cU_{\xi\xi} + cU_{\xi\eta}) + c(-cU_{\eta\xi} + cU_{\eta\eta})) \\
&= c^2 U_{\xi\xi} - 2c^2 U_{\xi\eta} + c^2 U_{\eta\eta} \, , \\
u_x &= U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta \, , \\
u_{xx} &= (U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x) + (U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x) \\
&= (U_{\xi\xi} + U_{\xi\eta}) + (U_{\eta\xi} + U_{\eta\eta}) \\
&= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} \, .
\end{aligned} \tag{29}$$

and next

$$U_{\xi\eta} = 0\,.$$

We integrate with respect to η ($U_{\xi\eta}$ does not depend on η)

$$U_{\xi} = \phi(\xi)\,.$$

We integrate with respect to ξ

$$U(\xi,\eta) = \int \phi(\xi) d\xi + G(\eta) = F(\xi) + G(\eta)\,.$$

Coming back to the old notation

$$\boxed{u(x,t) = F(x - ct) + G(x + ct)\,.} \tag{30}$$

Examples of the solutions of the wave equation:

$$u(x,t) = e^{x-ct},$$

$$u(x,t) = \sin(x+ct),$$

$$u(x,t) = (x+ct)^2 + e^{-(x-ct)^2}.$$

The first two equations represent the waves traveling to the left and to the right. The third equation is the combination of the waves traveling to the left and to the rigt.

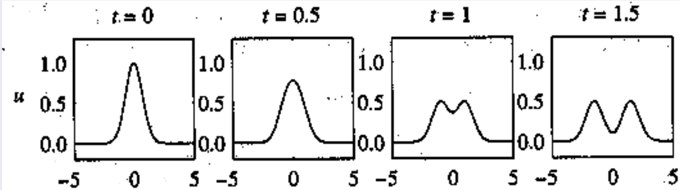

Fig. 11: Profiles of the solution of the wave equation with the initial profile $u(x,0) = e^{-x^2}$

We make the following assumptions: initial position $u(x, 0)$ and initial velocity $u_t(x, 0)$ are given for all x (e.g. $= 0$). The initial profile $u(x, 0) = f(x)$ and velocity $u_t(x, 0) = 0$.
We solve the following problem:

$$\begin{aligned} \text{PDE:} \quad & u_{tt} = c^2\, u_{xx}\,, \qquad -\infty < x < \infty\,,\ t > 0\,, \\ \text{IC:} \quad & u(x,0) = f(x)\,, \\ & u_t(x,0) = g(x)\,. \end{aligned}$$

We are looking for the solution in the general form:

$$u(x,t) = F(x-ct) + G(x+ct)\,.$$

We put the initial conditions for the posion

$$F(x) + G(x) = f(x)\,. \tag{31}$$

and for the velocity

$$-\,c\,F'(x) + c\,G'(x) = g(x)\,. \tag{32}$$

We divide by c and integrate from 0 to x

$$-F(x)+G(x)=-F(0)+G(0)+\frac{1}{c}\int_0^x g(s)ds\,. \qquad (33)$$

Equations (31) and (33) are the system of the linear equations for $F(x)$ and $G(x)$

$$F(x)=\frac{1}{2}f(x)-\frac{1}{2}(-F(0)+g(0)-\frac{1}{2c}\int_0^x g(s)ds\,,$$

$$G(x)=\frac{1}{2}f(x)-\frac{1}{2}(-F(0)+g(0)+\frac{1}{2c}\int_0^x g(s)ds\,,$$

The solution will have the form:

$$
\begin{aligned}
u(x,t) &= F(x-ct) + G(x+ct) = \\
&\frac{1}{2}f(x-ct) - \frac{1}{2}(-F(0)+G(0)) - \frac{1}{2c}\int_0^{x-ct} g(s)ds \\
&+\frac{1}{2}f(x+ct) - \frac{1}{2}(-F(0)+G(0)) - \frac{1}{2c}\int_0^{x+ct} g(s)ds \\
&= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+t) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds\,.
\end{aligned}
$$

Finally we obtain **the d'Alembert solution**

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds\,. \qquad (34)$$

of the wave equation. It is a very seldom case of the solution in the open form.

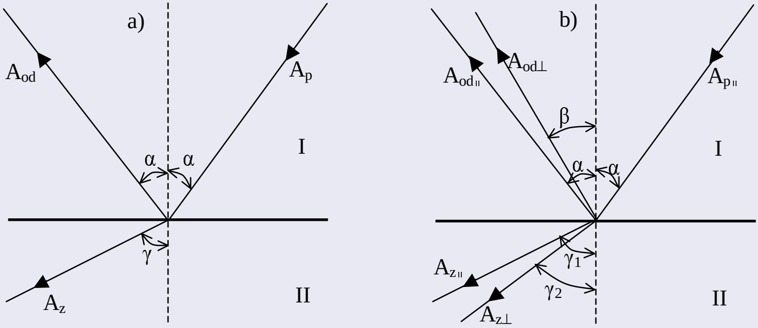

Fig. 12: Reflection and refraction of the wave in the wave medium (a), and in the elastic medium (b)

The end of the lecture 6